Going beyond Zeno through a pointwise asymptotically stable set in a hybrid system

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Introduction and outline

Going past Zeno:

- Hybrid inclusions, examples, and well-posedness
- Pointwise asymptotic stability
- Small ordinary time property
- Good behavior of limits of Zeno solutions
- Well-posedness past Zeno

Optimal control for pointwise asymptotic stability:

- Finite length Lyapunov functions
- Optimal and robust feedback stabilization in discrete time

Based on:

Set-valued Lyapunov functions for difference inclusions, G., Automatica 2011

Robustness of stability through necessary and sufficient Lyapunov-like conditions ..., G., SCL 2014

Results on optimal stabilization of a continuum of equilibria, G., CDC 2016

and joint work with R. Sanfelice:

Notions and sufficient conditions for pointwise asymptotic stability in hybrid systems, G. and Sanfelice, NOLCOS 2016

How well-posedness of hybrid systems can extend beyond Zeno times G. and Sanfelice, CDC 2016

Hybrid Inclusions

Hybrid Inclusions

A **hybrid inclusion** *combines a differential inclusion, a difference inclusion, and constraints on motions resulting from the inclusions.*

$$\begin{aligned} x \in C \qquad & \dot{x} \in F(x) \\ x \in D \qquad & x^+ \in G(x). \end{aligned}$$

Above, $\dot{x}$ *is velocity,* x^+ *is value after a jump.*

Single-valued case:

$$\begin{aligned} x \in C \qquad & \dot{x} = f(x) \\ x \in D \qquad & x^+ = g(x). \end{aligned}$$

Solutions:

parameterized by t and j, with (t, j) evolving in hybrid time domains;
satisfy $\phi(t, j) \in C$, $\dot{\phi}(t, j) \in F(\phi(t, j))$ when flowing;
satisfy $\phi(t, j) \in D$, $\phi(t, j + 1) \in G(\phi(t, j))$ when jumping.

Hybrid Dynamical Systems: Modeling, Stability, and Robustness G., Sanfelice, Teel, Princeton University Press, 2012

Pointwise asymptotic stability in a hybrid system — an example

Agents $z_1, z_2, \ldots, z_l \in \mathbb{R}^k$

agree on a target w *in the convex hull of* z_i*'s;*

converge exponentially to w*;*

every T *amount of time communicate and agree on a new* w*.*

Hybrid inclusion modeling this:

state $x = (z_1, z_2, \ldots, z_l, w, \tau)$;

if $\tau \geq 0$,

$$\dot{z}_i = w - z_i, \quad \dot{w} = 0, \quad \dot{\tau} = -1;$$

if $\tau = 0$,

$$z_i^+ = z_i, \quad w^+ \in \mathrm{con}\{z_1, z_2, \ldots, z_l\}, \quad \tau^+ = T.$$

Natural to expect convergence of z to and stability of the consensus set

$$\{z \,|\, z_1 = z_2 = \cdots = z_l\}$$

In fact, the following set is partially pointwise asymptotically stable:

$$A = \{(z, w) \,|\, z_1 = z_2 = \cdots = z_l = w\} \times [0, T]$$

Pointwise asymptotic stability in a hybrid system — a twisted example

Two agents $z_1, z_2 \in \mathbb{R}^k$

agree on a target $w = \dfrac{z_1 + z_2}{2}$;

converge to w *according to* $\dot{z}_i = c_i \dfrac{w - z_i}{\sqrt{|w - z_i|}}$, *where* $c_i > 0$.

update w *when one agent reduces its distance from* w *by a factor of* 4.

Hybrid inclusion with $x = (z_1, z_2, w, \tau) \in \mathbb{R}^{3k+1}$ and

$$C = \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \times [0, \infty),$$

$$F(x) = \left(c_1 \frac{w - z_1}{\sqrt{|w - z_1|}}, c_2 \frac{w - z_2}{\sqrt{|w - z_2|}}, 0, -1 \right),$$

$$D = \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \times \{0\},$$

$$G(x) = \left(z_1, z_2, \tfrac{z_1 + z_2}{2}, \min\{t_1, t_2\} \right) \text{ where } t_i := \frac{\sqrt{|a - z_i|}}{c_i}.$$

Then $A = \{z \mid z_1 = z_2 = w, \tau = 0\}$ is pointwise small ordinary time asymptotically stable.

(Nominal) well-posedness

(Nominal) well-posedness

For a differential equation $\dot{x} = f(x)$ or inclusion $\dot{x} \in F(x)$, if f or F is sufficiently regular, for every bounded sequence of solutions

- there exists a locally uniformly convergent subsequence (Arzela-Ascoli);
- the limit of the subsequence is a solution;

and if solutions are unique, this reduces to continuous dependence of solutions, over bounded time intervals, on initial conditions.

For a hybrid inclusion (C, F, D, G), under

Basic Assumptions: *C, D closed; F, G closed graph; $F(x)$ nonempty, convex for all $x \in C$; $G(x)$ nonempty for all $x \in D$.*

one has that, for every bounded sequence of solutions

- there exists a graphically convergent subsequence;
- the graphical limit of the subsequence is a solution;
- and more...

In short: (C, F, D, G) is well-posed.

(Nominal) well-posedness

Consequences of nominal well-posedness include:

(a) Solutions, over bounded hybrid time domains, depend on initial conditions in an outer-semicontinuous way and this can be characterized in terms of distances between graphs of solutions.

(b) Solutions, over bounded hybrid time domains, depend on initial conditions continuously when uniqueness of solutions can be ensured.

(c) The Krasovskii-LaSalle invariance principle, and other arguments relying on invariance, apply.

(d) For a compact asymptotically stable set the basin of attraction is open and from it, the convergence to the set is uniform and it admits a $\mathcal{KL}$ bound.

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However:

- Even if solutions have limits, the limits need not depend regularly on initial conditions.
- Infinite-horizon reachable sets need not depend regularly on initial conditions.
- Even if solutions are Zeno, the Zeno times need not depend regularly on initial conditions.

Pointwise asymptotic stability a.k.a. semistability

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Definition

The closed set A is *pointwise asymptotically stable* (PAS) if

- every point $a \in A$ is Lyapunov stable, that is, for every $a \in A$, $\varepsilon > 0$ there exists $\delta > 0$ such that every solution from $a + \delta\mathbb{B}$ remains in $a + \epsilon\mathbb{B}$;
- every solution is convergent and its limit is in A.

PAS is AS if A={a}, PAS $\implies$ AS if A compact, PAS ?? AS if A closed

PAS is not AS, even if every $a \in A$ is an equilibrium.

Standard Lyapunov conditions are not sufficient for PAS.

Examples:

- Steepest descent / negative gradient flow: $\dot{x} \in -\partial f(x)$ with f convex, $A = \arg\min f$
- Saddle-point dynamics: $\dot{x} \in -\partial_x h(x, y)$, $\dot{y} \in \partial_y h(x, y)$ with h convex-concave, A the set of saddle points
- Convex optimization algorithms (proximal-point, and more) with *Fejer property*: $\|x^+ - a\| \leq \|x - a\|$ for every $a \in A = \arg\min f$
- Numerous consensus algorithms, with $A = \{x \mid x_1 = x_2 = \cdots = x_n\}$

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Singular perturbation of autonomous linear systems, Campbell and Rose 1979
A continuous algorithm for finding the saddle points of convex-concave functions, Venets 84
Nontangency-based Lyapunov tests ..., Bhat, Bernstein 03
several articles by Haddad et al. 08, 09, 10,...
Arc-length-based Lyapunov tests ..., Bhat, Bernstein 10

links to consensus:

Stability of multiagent systems with time-dependent communication links, Moreau 05
Stability of leaderless discrete-time multi-agent systems, Angeli, Bliman 06

set-valued Lyapunov functions in discrete time

Set-valued Lyapunov functions for difference inclusions, G. 2011
Robustness of stability through necessary and sufficient Lyapunov-like conditions ..., G. 2014

Strict set-valued Lyapunov function for a closed set A

Definition

A set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a *strict set-valued Lyapunov function* if it has right growth and regularity properties and there exist continuous and positive definite with respect to A functions $c, d : \mathbb{R}^n \to \mathbb{R}$ so that

- for every solution $\phi : [0, T] \to \mathbb{R}^n$ to $\dot{x} \in F(x)$ such that $\phi(t) \in C$ for every $t \in (0, T)$,

$$W(\phi(t)) + \int_0^t c(\phi(s))\, ds\mathbb{B} \subset W(\phi(0)) \quad \forall t \in [0, T].$$

- $$W(G(x)) + d(x)\mathbb{B} \subset W(x) \quad \forall x \in D.$$

Compare to $V(G(x)) + d(x) \leq V(x)$

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Compare to $V(G(x)) + d(x) \leq V(x)$

Theorem

If there exists a strict set-valued Lyapunov function then A is PAS.

Converse results: exist for discrete time, expected for hybrid.

Weak set-valued Lyapunov function for a closed set A

Definition

A set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a *weak set-valued Lyapunov function* if it has right growth and regularity properties and

- for every solution $\phi : [0, T] \to \mathbb{R}^n$ to $\dot{x} \in F(x)$ such that $\phi(t) \in C$ for every $t \in (0, T)$,
$$W(\phi(t)) \subset W(\phi(0)) \quad \forall t \in [0, T].$$
- $$W(G(x)) \subset W(x) \quad \forall x \in D.$$

Weak set-valued Lyapunov function for a closed set A

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- for every solution $\phi : [0, T] \to \mathbb{R}^n$ to $\dot{x} \in F(x)$ such that $\phi(t) \in C$ for every $t \in (0, T)$,
$$W(\phi(t)) \subset W(\phi(0)) \quad \forall t \in [0, T].$$
- $$W(G(x)) \subset W(x) \quad \forall x \in D.$$

Theorem

If *there exists a continuous weak set-valued Lyapunov function W;*
every weakly invariant set on which W is constant is contained in A;
and C, F, D, G satisfies the Hybrid Basic Assumptions;
then A is PAS.

Proof: invariance principle.

Theorem

If C, F, D, G satisfies Basic Assumptions and the closed set $A \subset \mathbb{R}^n$ is PAS, then

(a) *the set-valued mapping L defined by*

$$L(x) = \bigcup \left\{ \lim_{t+j \to \infty} \phi(t,j) \,|\, \phi(0,0) = x \right\}$$

is outer semicontinuous and locally bounded;

(b) *the set-valued mapping $\overline{\mathcal{R}}_\infty$ defined by*

$$\overline{\mathcal{R}}_\infty(x) = \overline{\mathcal{R}_\infty(x)}$$

is outer semicontinuous and locally bounded and $\overline{\mathcal{R}}_\infty(x) = \mathcal{R}_\infty(x) \cup L(x)$ for every x, where the set-valued mapping $\mathcal{R}_\infty$ is defined by

$$\mathcal{R}_\infty(x) = \bigcup \{\phi(t,j) \,|\, \phi(0,0) = x, (t,j) \in \mathrm{dom}\phi\};$$

In single-valued case, outer semicontinuity and local boundedness is continuity.

Small ordinary time property

Definition

A set $A \subset \mathbb{R}^n$ is *pointwise small ordinary time asymptotically stable (PSOTAS)* if it is pointwise asymptotically stable and every $a \in A$ is SOT stable:

- for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathrm{length}_t \, \mathrm{dom}\, \phi < \varepsilon$ if $\phi(0,0) \in a + \delta\mathbb{B}$.

Theorem

Suppose that a closed set $A \subset \mathbb{R}^n$ is PAS and there exists a continuously differentiable $V : \mathbb{R}^n \to \mathbb{R}$ such that

(a) *V is positive definite with respect to A;*

(b) *there exists $c > 0$ and $\rho \in [0,1)$ such that*

$$\nabla V(x) \cdot f \leq -c\,(V(x))^\rho \quad \forall x \in C,\ f \in F(x);$$

(c) $V(g) \leq V(x) \quad \forall x \in D,\ g \in G(x);$

and there exist no nontrivial flowing solutions ϕ satisfying $\phi(t,j) \subset A$ for all $(t,j) \in \mathrm{dom}\,\phi$. Then A is PSOTAS.

The twisted example, revisited

Two agents $z_1, z_2 \in \mathbb{R}^k$

agree on a target $w = \dfrac{z_1 + z_2}{2}$;

converge to w *according to* $\dot{z}_i = c_i \dfrac{w - z_i}{\sqrt{|w - z_i|}}$, *where* $c_i > 0$.

update w *when one agent reduces its distance from* w *by a factor of* 4.

Consider the set $A = \{(z_1, z_2, w) \,|\, z_1 = z_2 = w\} \times \{0\}$. Then

- the set-valued mapping

$$W(\eta) = \mathrm{co}\{z_1, z_2, w\} \times [0, \max\{\tau, \min\{t_1, t_2\}\}]$$

  is a weak set-valued Lyapunov function;
- the functon $V(x) = \frac{1}{2}\left(\|z_1 - w\|^2 + \|z_2 - w\|^2\right)$ satisfies

$$\dot{V}(x(t)) \leq -2^{3/4} \min\{c_1, c_2\} \left(V(x(t))\right)^{3/4}.$$

PSOTAS of A follows!

Consequences of PSOTAS

Theorem

If C, F, D, G satisfies Hybrid Basic Assumptions and the closed set $A \subset \mathbb{R}^n$ is PSOTAS, then

(a) *every complete solution ϕ is Zeno: its Zeno time t_ϕ is finite*

$$t_\phi := \sup\{t \mid \exists j \ (t, j) \in \operatorname{dom} \phi\} < \infty;$$

(b) *for every graphically convergent ϕ_i to ϕ,*

$$\lim_{i \to \infty} t_{\phi_i} = t_\phi;$$

(c) *the function*

$$T(x) = \sup\{t_\phi \mid \phi(0,0) = x\}$$

is locally bounded, upper semicontinuous, and the supremum defining it is attained.

If solutions are unique, T is continuous.

Summary so far

Suppose that:

- C, F, D, G satisfies Hybrid Basic Assumptions;
- the closed set A is PSOTAS.

Then:

- limits of complete solutions depend reasonably on initial conditions;
- complete solutions are Zeno and their Zeno times depend reasonably on initial conditions.

Idea:

- solutions can be extended past their Zeno times, from their Zeno limits, via another hybrid system (not a new idea)
- so that their extensions depend reasonably on the pre-Zeno initial conditions!

Going past Zeno

Hybrid system 1:

- C_1, F_1, D_1, G_1 satisfies Basic Assumptions;
- the closed set A_1 is PSOTAS.

Re-initialization map:

- Ψ is locally bounded and has closed graph.

Hybrid system 2:

- C_2, F_2, D_2, G_2 satisfies Basic Assumptions;

Solution (ϕ, ψ), where:

- ϕ solves system 1;
- ψ solves system 2;
- $\psi(t_\phi, 0) \in \Psi\left(\lim_{t \to t_\phi, j \to \infty} \phi(t, j)\right)$, where t_ϕ is the Zeno time of ϕ.

Going past Zeno

Hybrid system 1:

- C_1, F_1, D_1, G_1 satisfies Basic Assumptions;
- the closed set A_1 is PSOTAS.

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Hybrid system 2:

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Solution (ϕ, ψ), where:

- ϕ solves system 1;
- ψ solves system 2;
- $\psi(t_\phi, 0) \in \Psi\left(\lim_{t \to t_\phi, j \to \infty} \phi(t, j)\right)$, where t_ϕ is the Zeno time of ϕ.

ψ <u>over bounded hybrid time domains</u> depends outer-semicontinuously on $\phi(0,0)$.

Consequence: if system 2 has a compact GAS set A_2, then convergence to A_2 is uniform from compact sets of $\phi(0,0)$, etc.

The twisted example, revisited

Hybrid system 1: two agents $z_1, z_2 \in \mathbb{R}^k$

agree on a target $w = (z_1 + z_2)/2$;

converge to w according to $\dot{z}_i = c_i(w - z_i)/\sqrt{|w - z_i|}$;

update w when one agent reduces its distance from w by a factor of 4.

Re-initialization map:

- Ψ is a continuous function.

Hybrid system 2:

- $\dot{x} = f_2(x)$, where f_2 is a Lipschitz continuous function.

Solution (ϕ, ψ), where:

- ϕ solves system 1;
- ψ solves system 2;
- $\psi(t_\phi, 0) = \Psi\left(\lim_{t \to t_\phi, j \to \infty} \phi(t, j)\right)$, where t_ϕ is the Zeno time of ϕ.

$\psi(t)$ depends continuously on $\phi(0,0)$, for $t > t_\phi$.

PAS from optimal control

PAS from optimal control — the big picture

If A is pointwise asymptotically controllable with locally exponential convergence rate for the hybrid inclusion with input

$$\begin{aligned} (x,u) \in C \qquad & \dot{x} \in F(x,u) \\ (x,u) \in D \qquad & x^+ \in G(x,u), \end{aligned}$$

then minimization of cost like

$$\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} d_A(\phi(t,j)) + \|u(t,j)\| \, dt + \sum_{j=0}^{J-1} d_A\left(\phi(t_{j+1},j)\right) + \|u(t_{j+1},j)\|$$

over infinite horizon yields optimal solutions that result in PAS.

Length-based sufficient condition for PAS in discrete time

Motivation:

Arc-length-based Lyapunov tests ..., Bhat, Bernstein 10

For $\dot{x} = f(x)$, *inequality* $\dot{V} \leq -\|f\|$ *implies* $\int_0^\infty \|\dot{x}(t)\|\, dt < \infty$... *finite length!*

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For $\dot{x} = f(x)$, inequality $\dot{V} \leq -\|f\|$ implies $\int_0^\infty \|\dot{x}(t)\|\, dt < \infty$... finite length!

Theorem

Suppose that

- *$A \subset \mathbb{R}^n$ is a closed set,*
- *$V : \mathbb{R}^n \to [0, \infty)$ is positive definite with respect to A and continuous at every $a \in A$,*
- *$\alpha : \mathbb{R}^n \to [0, \infty)$ is continuous and positive definite with respect to A.*

If

$$V(g(x)) + \alpha(x) + \|g(x) - x\| \leq V(x) \quad \forall x \in \mathbb{R}^n,$$

then A is pointwise asymptotically stable for $x^+ = g(x)$.

Proof idea:

- $V(g(x)) + \alpha(x) \leq V(x)$ ensures asymptotic stability of A (not pointwise!)
- $V(g(x)) + \|g(x) - x\| \leq V(x)$ ensures finite length, so convergence, of each solution
- $V(a) = 0$ at $a \in A$ and continuity implies small length of solutions from near a, and so Lyapunov stability of a

Pointwise asymptotic controllability in discrete time

Let $G : \mathbb{R}^n \times U \to \mathbb{R}^n$ be a function, $U \subset \mathbb{R}^k$ be a set. Consider

$$x^+ = G(x, u), \ u \in U.$$

The control system is pointwise asymptotically controllable to a set $A \subset \mathbb{R}^n$ with locally exponential convergence rate*:*

(a) *for every $\xi \in \mathbb{R}^n$, there exists an open-loop control $u_\xi : \{0, 1, 2, \dots\} \to U$ such that the resulting solution ϕ_ξ from ξ converges, and $\lim_{j\to\infty} \phi_\xi(j) \in A$;*

(b) *for every $a \in A$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every ξ with $\|\xi - a\| < \delta$, the solution ϕ_ξ from (a) is such that $\|\phi_\xi(j) - a\| < \varepsilon$ for every j;*

(c) *there exists an open set O containing A and constants $M, \gamma > 0$ such that, for every $\xi \in O$, the solution ϕ_ξ from (a) satisfies*

$$\|\phi_\xi(j) - a_\xi\| \le Me^{-\gamma j}\|\xi - a_\xi\| \quad \forall j,$$

where $a_\xi = \lim_{j\to\infty} \phi_\xi(j)$.

Optimal feedback stabilization in discrete time

Theorem

If G is continuous, U compact, A closed and PAC Assumption holds for

$$x^+ = G(x, u), \;\; u \in U,$$

then there exists a feedback $u : \mathbb{R}^n \to U$ such that A is robustly PAS for

$$x^+ = G(x, u(x)).$$

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Proof idea: let $d_A(x)$ be the distance from A: $d_A(x) = \min_{a \in A} \|x - a\|$, let

$$V(\xi) \;\; = \;\; \inf_{u:\{0,1,2,\dots\} \to U} \sum_{j=0}^{\infty} d_A(\phi(j)) + \|\phi(j+1) - \phi(j)\|$$

and consider optimal (or suboptimal) feedback.

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and consider optimal (or suboptimal) feedback.

Under further assumptions on g, for example $\|G(x, u) - x\| \leq \nu(d_A(x) + \|u\|)$, can consider

$$V(\xi) \;\; = \;\; \inf_{u:\{0,1,2,\dots\} \to U} \sum_{j=0}^{\infty} d_A(\phi(j)) + \|u(j)\|$$

Definition

A is *robustly PAS* for

$$x^+ = G(x, u(x)) =: g(x)$$

if there exists $\rho : \mathbb{R}^n \to [0, \infty)$, continuous and positive definite with respect to A and such that A is PAS for

$$x^+ \in g_\rho(x),$$

where $g_\rho : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping defined by $g_\rho(x) = \bigcup_{y \in g(x+\rho(x)\mathbb{B})} y + \rho(y)\mathbb{B}$.

This covers measurement error, actuator error, and external disturbance.

Given the optimal value function V, define a set-valued mapping W:

$$W(x) = x + V(x)\mathbb{B}.$$

This W is a continuous set-valued Lyapunov function, and a result of

Robustness of stability through necessary and sufficient Lyapunov-like conditions ..., G. 2014

implies robustness.

The End

Summary:

- In the right circumstances (PSOTAS), Zeno solutions to hybrid systems can be extended past their Zeno times and well-posedness is preserved.
- PAS can be achieved via optimal control.

Further questions for hybrid systems:

- Converse set-valued Lyapunov results and relation to robustness of PAS?
- Infinitesimal characterization of set-valued Lyapunov functions (in continuous and then hybrid time)?

Thank you for your attention.

Bonus: Finite-length Lyapunov function in hybrid setting

Definition

A continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$ is a *finite-length Lyapunov function* if V is positive definite with respect to A, and there exist continuous $c, d : \mathbb{R}^n \to [0, \infty)$, positive definite with respect to A, such that the following hold:

for every $x \in C$, $f \in F(x)$, $$\nabla V(x) \cdot f \leq -c(x) - \|f\|,$$

- for every $x \in D$, $g \in G(x)$,
$$V(g) - V(x) \leq -d(x) - \|g - x\|.$$

Inspired by *Arc-length-based Lyapunov tests ...*, Bhat, Bernstein 10
which used $\nabla V(x) \cdot f(x) \leq -\|f(x)\|$ for $\dot{x} = f(x)$

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$$V(g) - V(x) \leq -d(x) - \|g - x\|.$$

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Theorem

If there exists a finite-time Lyapunov function then A is PAS.

Key to proof / justification of name:

$$\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} \|\dot{\phi}(t,j))\| \, ds + \sum_{j=1}^{J} \|\phi(t_j, j+1) - \phi(t_j, j)\| \leq V(\phi(0,0)) - V(\phi(T,J))$$