Hysteresis and Hybrid Systems

OptHySYS
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Hysteresis

Input-output relationship between time-dependent (scalar) quantities which presents a particular type of memory.

- We are in presence of hysteresis when the memory is “rate-independent”:
- The behavior of the output $w$ depends only on the “history” of the input $u$ (the sequence of reached values) and not on how fast $u$ runs its history (the time-derivative).
Hysteresis cycle
In particular, the operator \( \mathcal{F} \) is causal (i.e. \( \mathcal{F}[u](t) \) depends only on \( u_{[0,t]} \)) it is not linear, it is not differentiable, \( w(\cdot) \) may be discontinuous, \( \mathcal{F}[u \circ \varphi] = \mathcal{F}[u] \circ \varphi \), for any (positive) time-scaling \( \varphi \), and typically the relationship \( u \mapsto w = \mathcal{F}[u(\cdot)] \) is not a "differential" relationship (even if sometimes this is "almost" true).
The Play operator
The Play operator
The Play operator
The Play operator
The Play operator
The Play operator
The Play operator
The Play operator
The Play operator
Given $u$, we cannot know the value of $w$ without knowing the past evolution of $u$. 

The Play operator
The Play operator

$|u - w| < a$

$u - w = a$

$u - w = -a$
The Play operator

Variational inequality
\[
\begin{cases}
w'(t)(u(t) - w(t) - v) \geq 0 & \forall \ |v| \leq a \quad \text{for almost every } t \\
|u(t) - w(t)| \leq a & \forall t
\end{cases}
\]

Discontinuous ODE
\[
w'(t) = \chi_{\{u-w=a\}}(t)(u'(t))^+ - \chi_{\{u-w=-a\}}(t)(u'(t))^- \quad \text{for almost every } t
\]

\[
\chi_{\{u-w=a\}}(t) = \begin{cases}
1 & \text{if } u(t) - w(t) = a \\
0 & \text{otherwise}
\end{cases}
\]

\[
(u'(t))^+ = \max\{u'(t),0\}
\]

\[
(u'(t))^- = \max\{-u'(t),0\}
\]
Hybrid Systems formulation

• ..\Hybrid\hybrid_short.pdf
ODE with hysteresis

\[
\begin{align*}
  y'(t) &= f(y(t), w(t)), \quad t > 0, \\
  w(t) &= \mathcal{F}[y(\cdot)](t), \quad t > 0, \\
  y(0) &= x \\
  w(0) &= w^0
\end{align*}
\]
Play as hybrid system

\[
\begin{aligned}
\begin{cases}
y' = f(y, w) \\
w = \mathcal{F}[y]
\end{cases}
\end{aligned}
\]

\(\mathcal{F}\) is the Play operator

\[
D = \{(y, w, z) \text{ s.t. } |y - w| \leq a, z = 0\},
\]

\[
C_1 = \{(y, w, z) \text{ s.t. } y - w \geq a, z \in \{0,1\}\},
\]

\[
C_{-1} = \{(y, w, z) \text{ s.t. } y - w \leq -a, z \in \{0,-1\}\}
\]

\[
\begin{cases}
(y, w, z)' = (f(y, w), z | f(y, w)|, 0) & \text{if } (y, w, z) \in D \\
(y, w, z)^+ = (y, w, z^+) = (y, w, g(z)) & \text{if } (y, w, z) \in C
\end{cases}
\]
Switching hysteresis
(thermostat or delayed relay)
Switching evolution of output subject to the continuous input $u$
Switching hysteresis
(thermostat or delayed relay)
Delayed thermostat

\[
\begin{align*}
    w(t) = & \begin{cases} 
        w(0) & \text{if } X_t := \{ \tau \in [0, t] \text{ s.t. } u(t) = \rho_1 \text{ or } u(t) = \rho_2 \} = \emptyset \\
        -1 & \text{if } u(\max X_t) = \rho_1 \\
        1 & \text{if } u(\max X_t) = \rho_2 
    \end{cases} \\
\end{align*}
\]

A variational principle:
given the input \(u\), the output \(w\) minimizes, for every \(t\),
the total variation (number of jumps) on \([0, t]\),
among all functions \(\omega\) such that
\[
(u(\tau), \omega(\tau)) \in \left( (-\infty, \rho_2] \times \{-1\} \right) \cup \left( [\rho_1, +\infty[ \times \{1\} \right)
\]
Delayed thermostat as hybrid system

\[
\begin{cases}
y' = f(y, w) \\
w = \mathcal{F}[y]
\end{cases}
\]

\(\mathcal{F}\) is the delayed thermostat

\[D = \left( (-\infty, \rho_2] \times \{-1\} \right) \cup \left( [\rho_1, +\infty) \times \{1\} \right)\]

\[C = \left( [\rho_2, +\infty) \times \{-1\} \right) \cup \left( (-\infty, \rho_1] \times \{1\} \right)\]

\[\begin{cases}
(y, w)' = (f(y, w), 0) & \text{if } (y, w) \in D \\
(y, w)^+ = (y, w^+) = (y, g(w)) & \text{if } (y, w) \in C
\end{cases}\]
Optimal control problem

\[
\begin{aligned}
\dot{y}(t) &= f(y(t), w(t), \alpha(t)), \quad t > 0, \\
w(t) &= \mathcal{F}[y(\cdot)](t), \quad t > 0, \\
y(0) &= x, \quad w(0) = w^0
\end{aligned}
\]

\[\alpha : [0, +\infty[ \rightarrow A, \text{ measurable control, } A \subset \mathbb{R}^p \text{ compact}\]

Infinite horizon cost

\[J(x, w^0, \alpha) = \int_{0}^{+\infty} e^{-\lambda t} \ell(y(t), w(t), \alpha(t)) dt\]
Optimal control problem

\[ V(x, w^0) = \inf_{\alpha} J(x, w^0, \alpha), \text{ value function} \]

The aim is, using Dynamic Programming, to characterize the value function as the unique (viscosity) solution of a suitable “Hamilton-Jacobi-Bellman problem”.
Optimal control problem

• The optimal control problem of systems intrinsically exhibiting hysteresis is a very important feature.

• Sometimes, an artificial hysteresis is introduced in the mathematical model in order to approximate some irregularities of the data, such as discontinuities and others.

• In the following two examples of “artificially introduced hysteresis” are given.
Discontinuities on the data.

\[ y' = f_1(y, \alpha), \quad \ell_1(y, \alpha) \quad \text{at} \quad 0 \quad \text{and} \quad y' = f_2(y, \alpha), \quad \ell_2(y, \alpha) \]
\[ J(x, \alpha) = \int_0^\infty e^{-\lambda t} \ell_i (y_x(t, \alpha), \alpha(t)) \, dt \]

\[ V(x) = \inf_{\alpha \in A} J(x, \alpha) \quad \forall x \in R \]
\[\lambda V(x) + H_1(x, \nabla V(x)) = \lambda V(x) + \max_a \{- f_1(x, a) \cdot \nabla V(x) - \ell_1(x, a)\} = 0\]

Two different Hamilton-Jacobi equations on the two half lines. Which condition on the junction \(x=0\), in order to characterize the value function as the unique solution?
Discontinuous dynamics

$f_{-1}$

$f_1$
Discontinuous dynamics

$f_1$

$f_{-1}$
Sliding Modes

Filippov trajectories, selection theorem...

\[ \mu f_1(x, a_1) + (1 - \mu) f_2(x, a_2) \]
\[ \mu \in [0,1] \]
Regular tangential dynamics
Singular tangential dynamics
Discontinuous dynamics
Discontinuous dynamics

\[ J(x, \alpha) = \int_{0}^{\infty} e^{-\lambda t} \ell_i (y_x(t, \alpha), \alpha(t)) \, dt \]

\[ V(x) = \inf_{\alpha \in A} J(x, \alpha) \quad \forall \, x \in \mathbb{R} \]
Hamilton-Jacobi on multi-domains

\[ \lambda V(x) + \max_{a \in A} \left\{ -f_2(x, a) \cdot \nabla V(x) - \ell_2(x, a) \right\} = \lambda V(x) + H_2(x, \nabla V(x)) = 0 \]

Problem:
determine a suitable Hamilton-Jacobi problem on \( R^n \) as well as "restricted to \( \Gamma \)"
such that it has a unique solution \( V \) on \( R^n \)
and to give a meaning to \( V \)
as value function of suitable optimal control problem

\[ \lambda V(x) + \max_{a \in A} \left\{ -f_1(x, a) \cdot \nabla V(x) - \ell_1(x, a) \right\} = \lambda V(x) + H_1(x, \nabla V(x)) = 0 \]
One recent result in literature

Under suitable controllability hypotheses

the maximal subsolution $U^+$ of the following problem

$$
\begin{align*}
& \lambda V + H_2(x, \nabla V) = 0 \quad \text{in } \left( \mathbb{R}^n \setminus \Gamma \right)_2 \\
& \lambda V + H_{\Gamma}^{\text{reg}}(x, \nabla V) = 0 \quad \text{on } \Gamma \\
& \lambda V + H_1(x, \nabla V) = 0 \quad \text{in } \left( \mathbb{R}^n \setminus \Gamma \right)_1 \\
& \max \{ \lambda V + H_1, \lambda V + H_2 \} \geq 0 \quad \text{on } \Gamma \\
& \min \{ \lambda V + H_1, \lambda V + H_2 \} \leq 0 \quad \text{on } \Gamma
\end{align*}
$$

where $H_{\Gamma}^{\text{reg}}$ is the tangential Hamiltonian on $\Gamma$ taking into account only there regular tangential trajectories, is the value function of the optimal control problem with only regular tangential dynamics.
An hybrid thermostatic approximation of the discontinuity
Hybrid thermostatic approximation
The evolution of $(y,z)$

- Governed by $f_2(\cdot, \alpha)$
- Governed by $f_1(\cdot, \alpha)$

The new state variable is the pair $(x, z) \in \mathbb{R}^n \times \{1, 2\} = \mathbb{R}^n \times \{\text{blue, red}\}$
We can think of our problem as a coupling of two problems of exit-time type which mutually exchange their exit-costs (boundary conditions)

\[
\begin{align*}
\mathcal{L}V_\varepsilon(x,1) + \max_{a \in A} \left\{ - f_1(x,a) \cdot \nabla V_\varepsilon(x,1) - \ell_1(x,a) \right\} &= 0 \quad \text{in } \Omega_1^\varepsilon \\
V_\varepsilon(x,1) &= V_\varepsilon(x,2) \quad \text{on } \partial \Omega_1^\varepsilon \\
\mathcal{L}V_\varepsilon(x,2) + \max_{a \in A} \left\{ - f_2(x,a) \cdot \nabla V_\varepsilon(x,-1) - \ell_2(x,a) \right\} &= 0 \quad \text{in } \Omega_2^\varepsilon \\
V_\varepsilon(x,2) &= V_\varepsilon(x,1) \quad \text{on } \partial \Omega_2^\varepsilon
\end{align*}
\]

In series of papers it is proven that the value function of this (and others) switching thermostatic problem is the unique viscosity solution of the coupled exit-time problem.
What happens when $\varepsilon \to 0$?
Obviously, we recover a situation as the multi-domains situation.
Theorem: Let \( \tilde{V}_\varepsilon : \mathbb{R}^2 \setminus \{ x = 0 \} \to \mathbb{R} \) be
\[
\tilde{V}_\varepsilon = \begin{cases} 
V_\varepsilon(x, y, 1) & \text{if } x \geq 0, y \in \mathbb{R} \\
V_\varepsilon(x, y, 2) & \text{if } x \leq 0, y \in \mathbb{R}
\end{cases}
\]
As \( \varepsilon \to 0 \), the sequence of function \( \tilde{V}_\varepsilon \) uniformly converges on \( \mathbb{R}^2 \setminus \{ x = 0 \} \) to a continuous function \( \tilde{V} \).
Moreover, also due to controllability, such a limit function uniquely extends to the line \( \{ x = 0 \} \) and it satisfies the problem
\[
\begin{align*}
\lambda V + H_2(x, \nabla V) &= 0 \text{ in } (\mathbb{R}^2 \setminus \Gamma)_2 \\
\lambda V + H_{\Gamma}^{reg}(x, \nabla V) &= 0 \text{ on } \Gamma \\
\lambda V + H_1(x, \nabla V) &= 0 \text{ in } (\mathbb{R}^2 \setminus \Gamma)_1 \\
\max\{\lambda V + H_1, \lambda V + H_2\} &\geq 0 \text{ on } \Gamma \\
\min\{\lambda V + H_1, \lambda V + H_2\} &\leq 0 \text{ on } \Gamma
\end{align*}
\]
Our guess is that \( \tilde{V} \) coincides with the value function \( U^+ \) of the regular control problem, that is it is the maximal subsolution.
A one-dimensional three-junction problem
Optimal visiting problem

Problem: visit three sites minimizing time, with evolution subject to
\[ y'(t) = f(y(t), \alpha(t)), \quad y(0) = x \]

\[ t_\alpha(x) = \inf \{ t \geq 0 | \forall i \in \{1,2,3\} \exists 0 \leq t_i \leq t, y(t_i) \in \mathcal{T}_i \} \]

\[ T(x) = \inf_{\alpha} t_\alpha(x) \quad \text{optimal visiting function} \]
• The problem is obviously reminiscent of the famous **Traveling Salesman Problem**: minimizing the length of the path for passing through \( m \) cities.

• It is then characterized by a **high computational complexity**: many sub-problems must be addressed before solving the initial problem.
$\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$
$T_1$ minimum time function for reaching $\mathcal{T}_1$, which solves
\[
\begin{cases}
\sup_a \{ - f(x, a) \cdot \nabla T(x) \} = 1 \\
T = 0 \text{ on } \partial \mathcal{T}_1
\end{cases}
\]
$T_2$ minimum time function for reaching $\mathcal{T}_2$, which solves

$$\left\{ \begin{array}{l}
\sup_{a} \{- f(x, a) \cdot \nabla T(x)\} = 1 \\
T = 0 \text{ on } \partial \mathcal{T}_2
\end{array} \right.$$
$T_3$ minimum time function for reaching $\mathcal{T}_3$, which solves
\[
\left\{ \begin{array}{l}
\sup_{a} \left\{ - f(x, a) \cdot \nabla T(x) \right\} = 1 \\
T = 0 \text{ on } \partial \mathcal{T}_3
\end{array} \right.
\]
$T_{1,2}$ optimal visiting function for reaching $\mathcal{T}_1$ and $\mathcal{T}_2$, which solves

$$\left\{ \sup_{a} \{-f(x,a) \cdot \nabla T(x)\} = 1 \right\}$$

$$\begin{cases} 
T = T_2 & \text{on } \partial \mathcal{T}_1 \\
T = T_1 & \text{on } \partial \mathcal{T}_2 
\end{cases}$$
\( T_{1,3} \) optimal visiting function for reaching \( \mathcal{T}_1 \) e \( \mathcal{T}_3 \), which solves

\[
\begin{align*}
\sup_{a} \{ -f(x, a) \cdot \nabla T(x) \} &= 1 \\
T &= T_3 \quad \text{on} \quad \partial \mathcal{T}_1 \\
T &= T_1 \quad \text{on} \quad \partial \mathcal{T}_3
\end{align*}
\]
$T_{2,3}$ optimal visiting function for reaching $\mathcal{T}_2$ e $\mathcal{T}_3$, which solves

$$\sup_a \{- f(x,a) \cdot \nabla T(x)\} = 1$$

$$T = T_3 \text{ on } \partial \mathcal{T}_2$$

$$T = T_2 \text{ on } \partial \mathcal{T}_3$$
$T_{1,2,3}$ optimal visiting function for reaching $T_1$, $T_2$, and $T_3$, which solves
\[
\begin{cases}
\sup_{a} \{- f(x,a) \cdot \nabla T(x)\} = 1 \\
T = T_{2,3} \text{ on } \partial T_1 \\
T = T_{1,3} \text{ on } \partial T_2 \\
T = T_{1,2} \text{ on } \partial T_3
\end{cases}
\]

$m=3 \Rightarrow 7=2^m-1$ sub-problems
Goal

• Use Dynamic Programming for writing a “single” equation uniquely satisfied by the optimal visiting function.

• An immediate problem:

• The Dynamic Programming Principle does not hold.

• “Pieces of optimal trajectories are not optimal”!
Optimal trajectory for $x$.

Optimal for $y(t)$ too.
Optimal trajectory for $x$

Optimal for $y(t)$ too.
Optimal trajectory for $x$

But not for $y(\tau)!$
Optimal trajectory for $x$

But not for $y(\tau)$!
Need of memory

• We need a sort of memory!
• We have to keep in mind whether the $i$-th target is already visited or not.
• For every $i$, we need a positive scalar $w_i$, evolving in time, which is zero if and only if we have already reached the $i$-th target.
Hysteresis operator

\[ u_i(t) = \text{dist}(y(t), T_i), \quad w_i(t) = \min_{\tau \in [0,t]} \left( \text{dist}(y(\tau), T_i) \right) = \min_{\tau \in [0,t]} u_i(\tau) \]
Hysteresis operator

\[
\begin{align*}
  u_i(t) &= \text{dist}(y(t), \mathcal{T}_i), &
  w_i(t) &= \min_{\tau \in [0,t]} \left( \text{dist}(y(\tau), \mathcal{T}_i) \right) = \min_{\tau \in [0,t]} u_i(\tau)
\end{align*}
\]
Hysteresis operator

\[ u_i(t) = \text{dist}(y(t), T_i), \quad w_i(t) = \min_{\tau \in [0,t]} \left( \text{dist}(y(\tau), T_i) \right) = \min_{\tau \in [0,t]} u_i(\tau) \]
Hysteresis operator

\[ u_i(t) = dist(y(t), T_i), \quad w_i(t) = \min_{\tau \in [0,t]} \left( dist(y(\tau), T_i) \right) = \min_{\tau \in [0,t]} u_i(\tau) \]

\( w_i \) can be written as the unique solution of a discontinuous ODE depending on \( u_i \) as parameter; or as solution of variational inequality, also depending on \( u_i \).
Need of memory

• We need a sort of memory!
• We have to keep in mind whether the $i$-th target is already visited or not.
• For every $i$, we need a positive scalar $w_i$, evolving in time, which is zero if and only if we have already reached the $i$-th target.
• The new state variable is $(x, w_1, w_2, ..., w_m)$ and the new target is \{w_1=w_2=...=w_m=0\}.
• We recover the Dynamic Programming Principle and we can write a (discontinuous) Hamilton-Jacobi equation uniquely satisfied by the optimal visiting function.
• We pay a price: we add more variables (the $w_i$) and the equation is discontinuous.
• A result in this sense was already obtained in the case when the sequence of visited targets is free and at the end it is not mandatory to return to the initial state (as indeed is for the classical Salesman Travelling Problem)