Hysteresis and Hybrid Systems

OptHySYS

eastside seminar

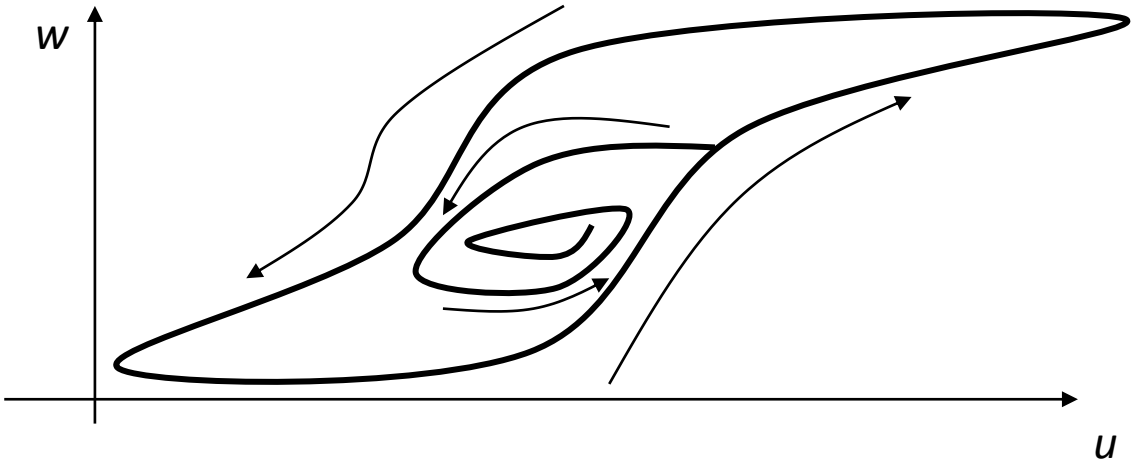
June 3rd 2015

Hysteresis

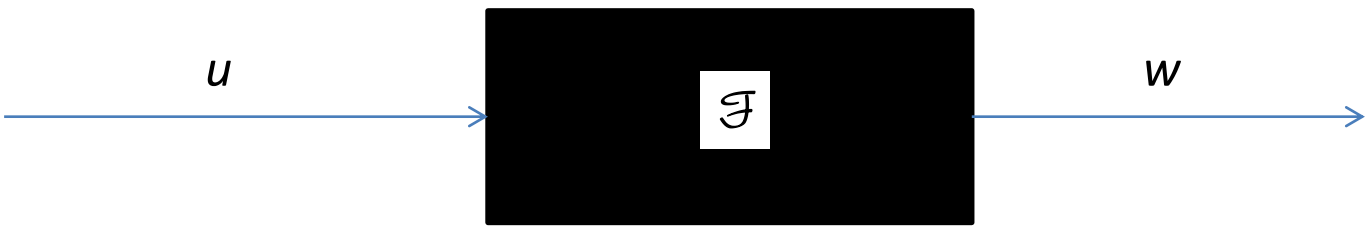
Input-output relationship between time-dependent (scalar) quantities which presents a particular type of memory.

- We are in presence of hysteresis when the memory is “rate-independent”:
- The behavior of the output *w* depends only on the “history” of the input *u* (the sequence of reached values) and not on how fast *u* runs its history (the time-derivative).

Hysteresis cycle



Hysteresis



$w(t)=\mathcal{F}[u(.)](t)$

In particular, the operator $\mathcal{F}$ is causal (i.e. $\mathcal{F}[u](t)$ depends only on $u_{|[0,t]}$)

it is not linear,

it is not differentiable,

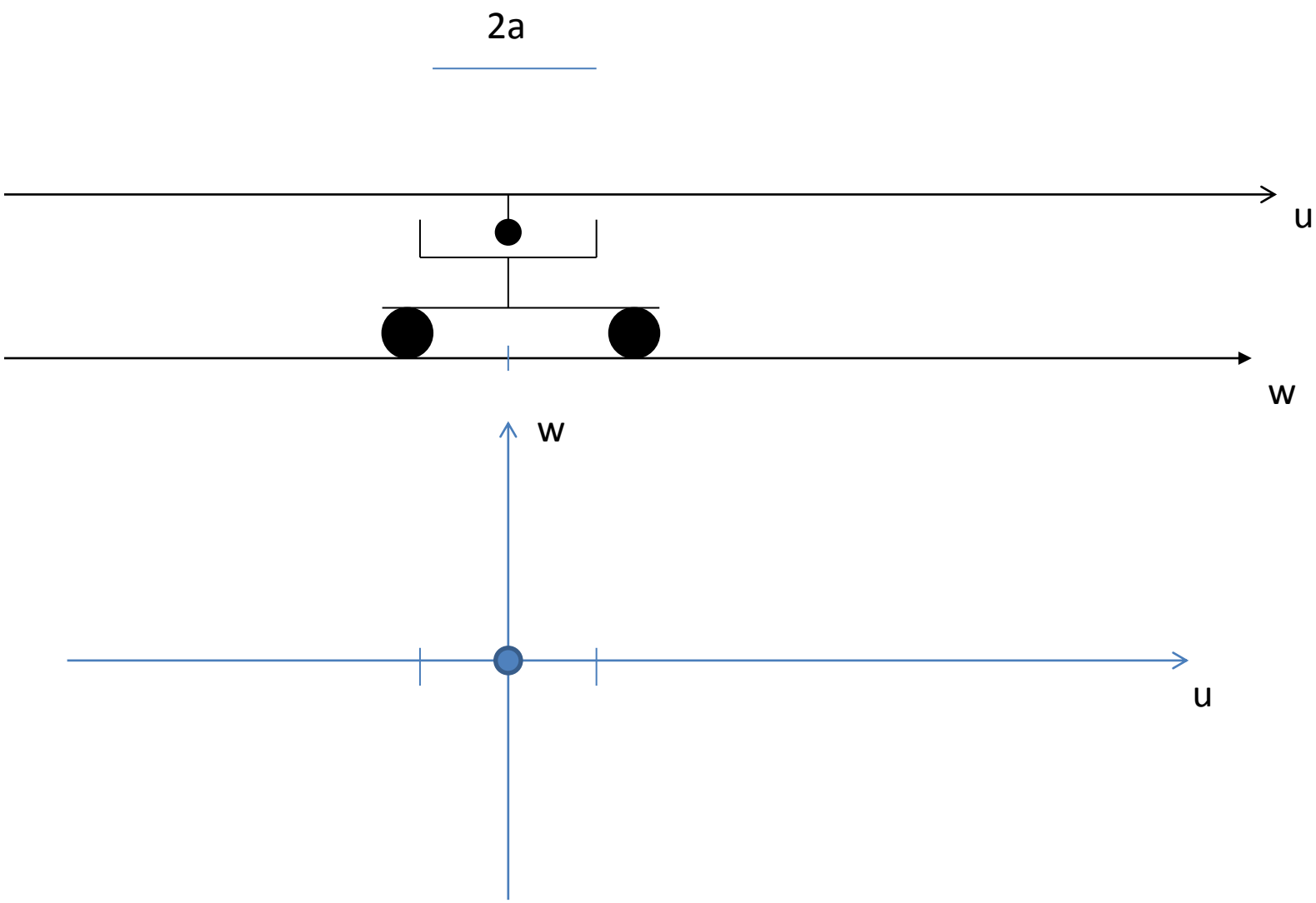
$w(\cdot)$ may be discontinuous,

$\mathcal{F}[u \circ \varphi] = \mathcal{F}[u] \circ \varphi$, for any (positive) time-scaling φ,

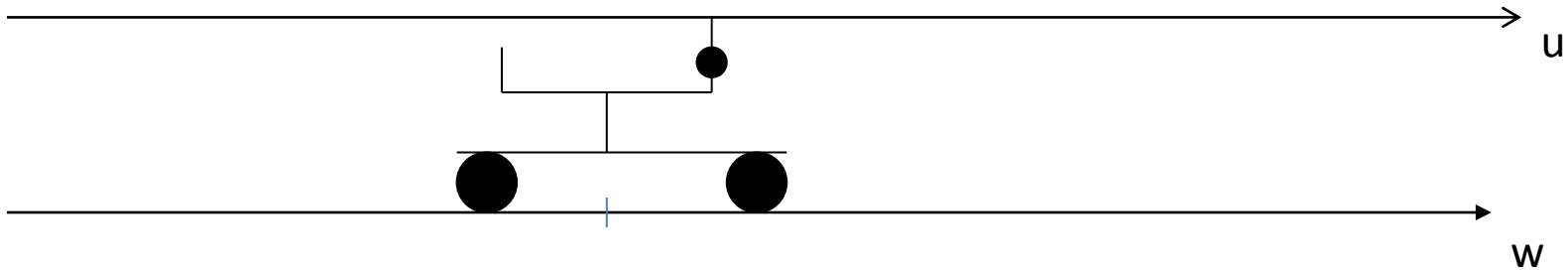
and tipically the relationship $u \mapsto w = \mathcal{F}[u(\cdot)]$ is not a "differential" relationship

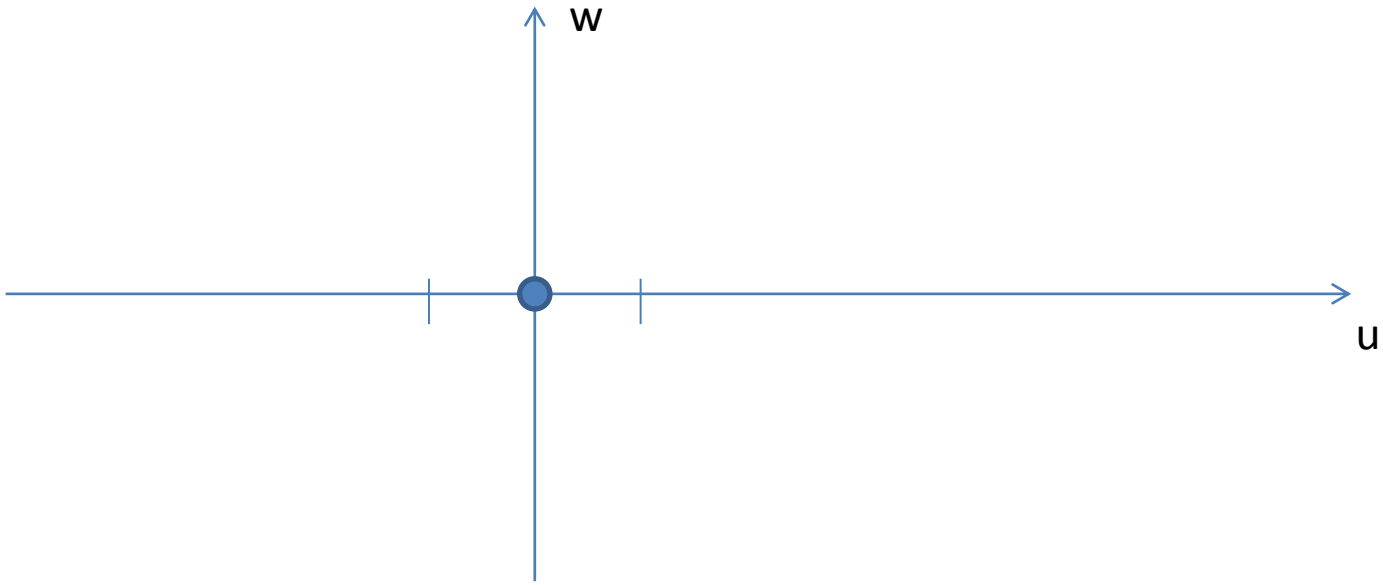
(even if sometimes this is "almost" true)

The Play operator



The Play operator





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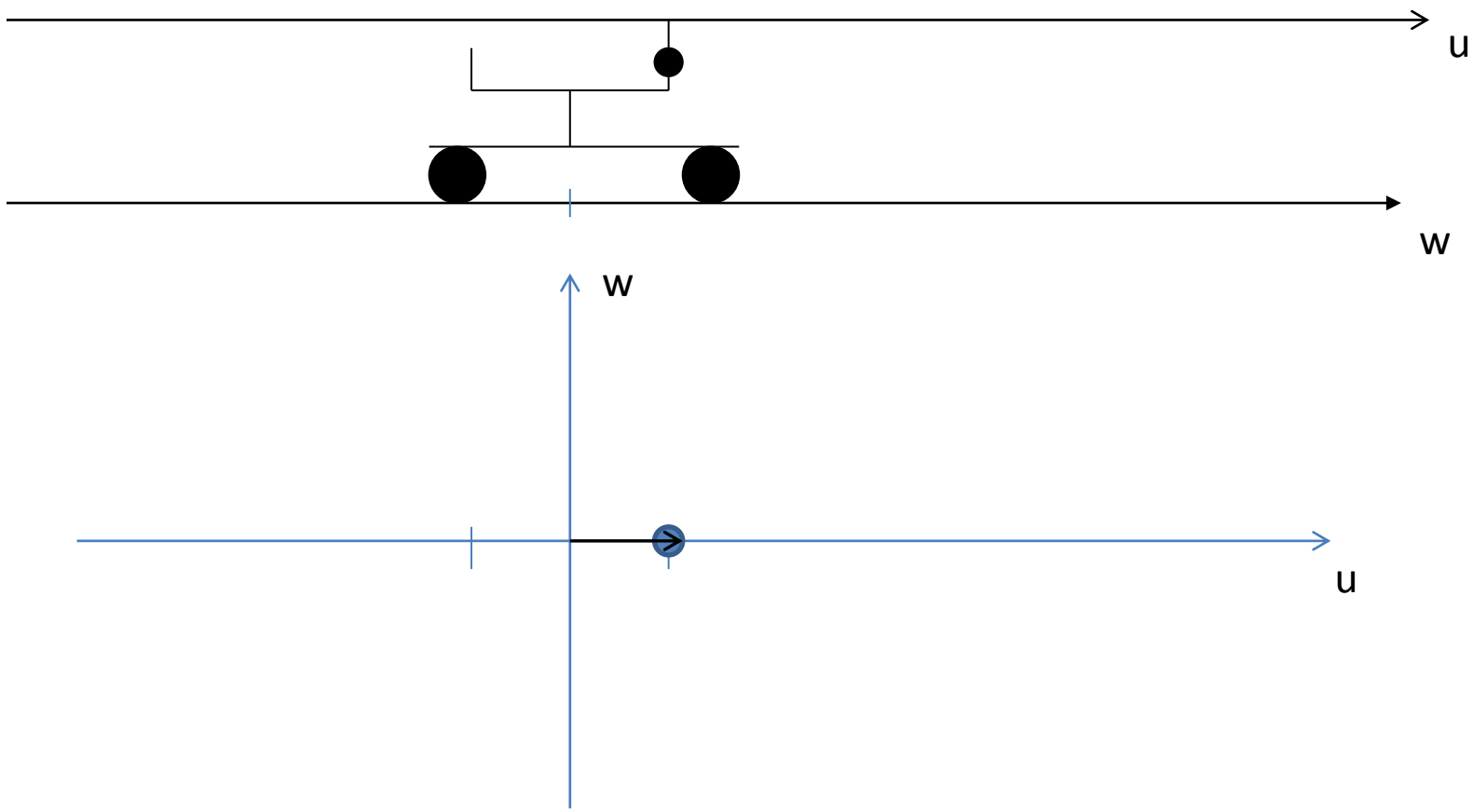

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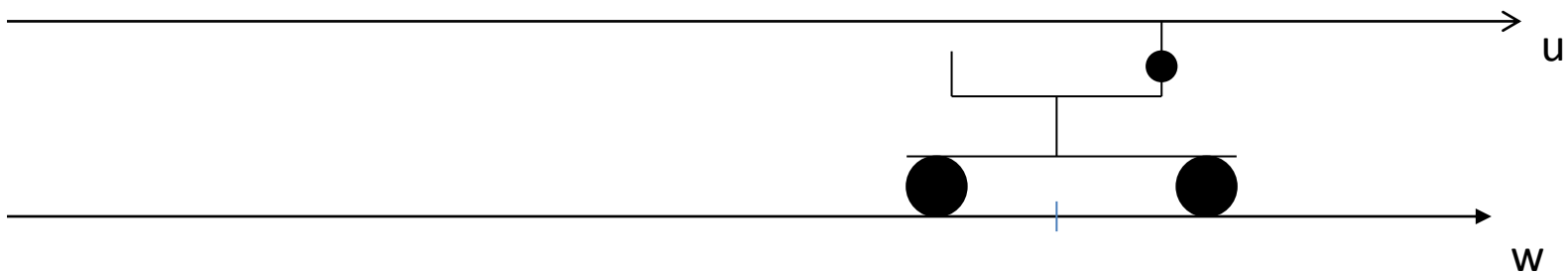

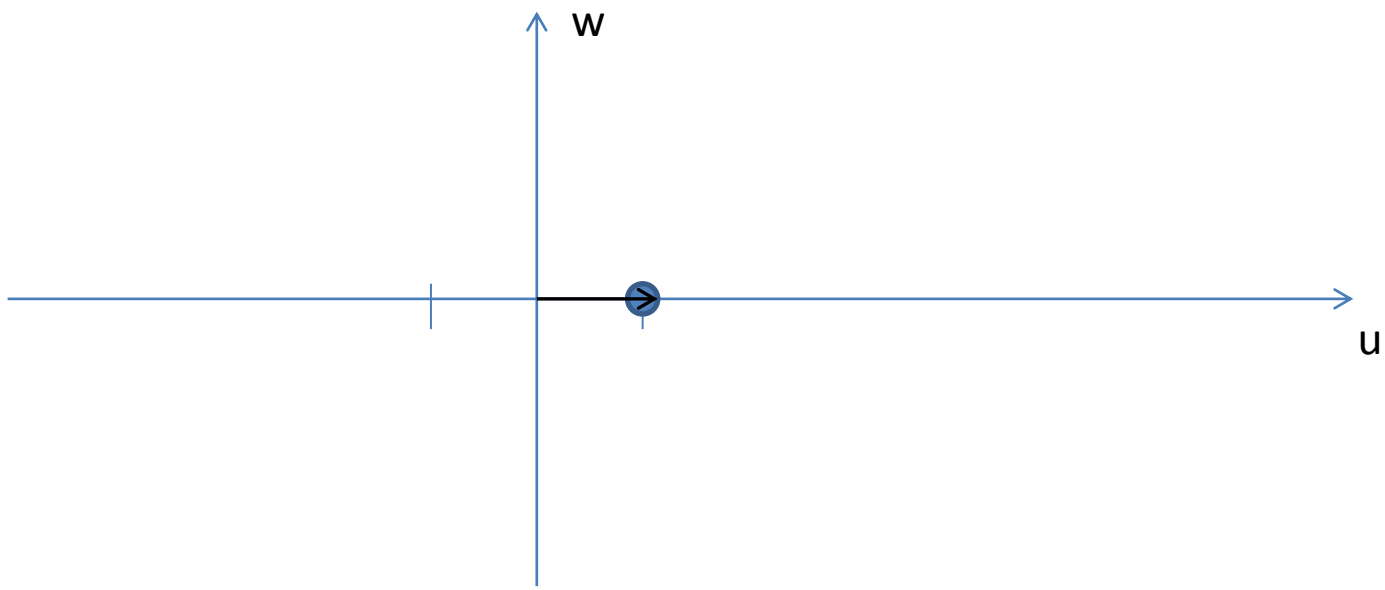

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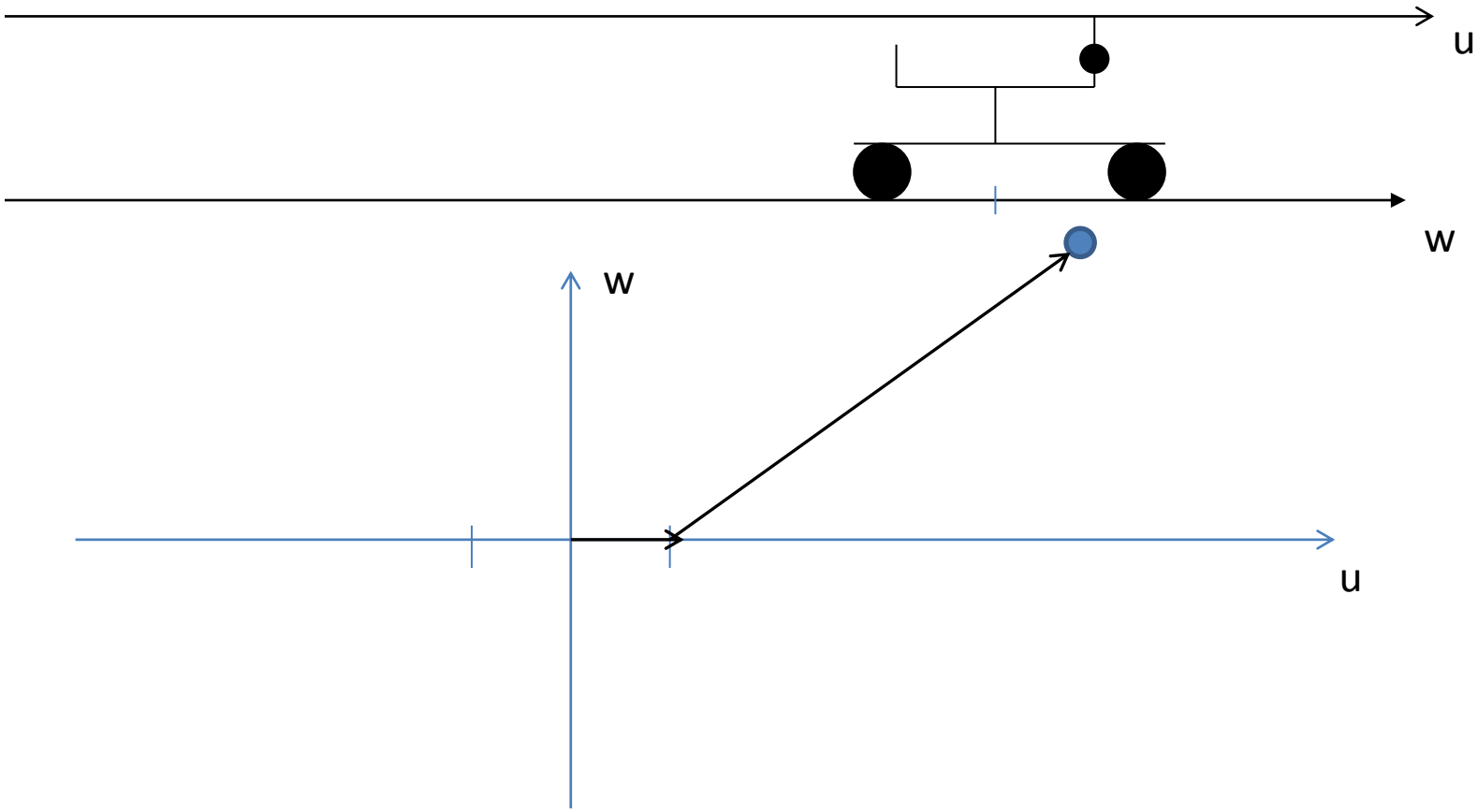

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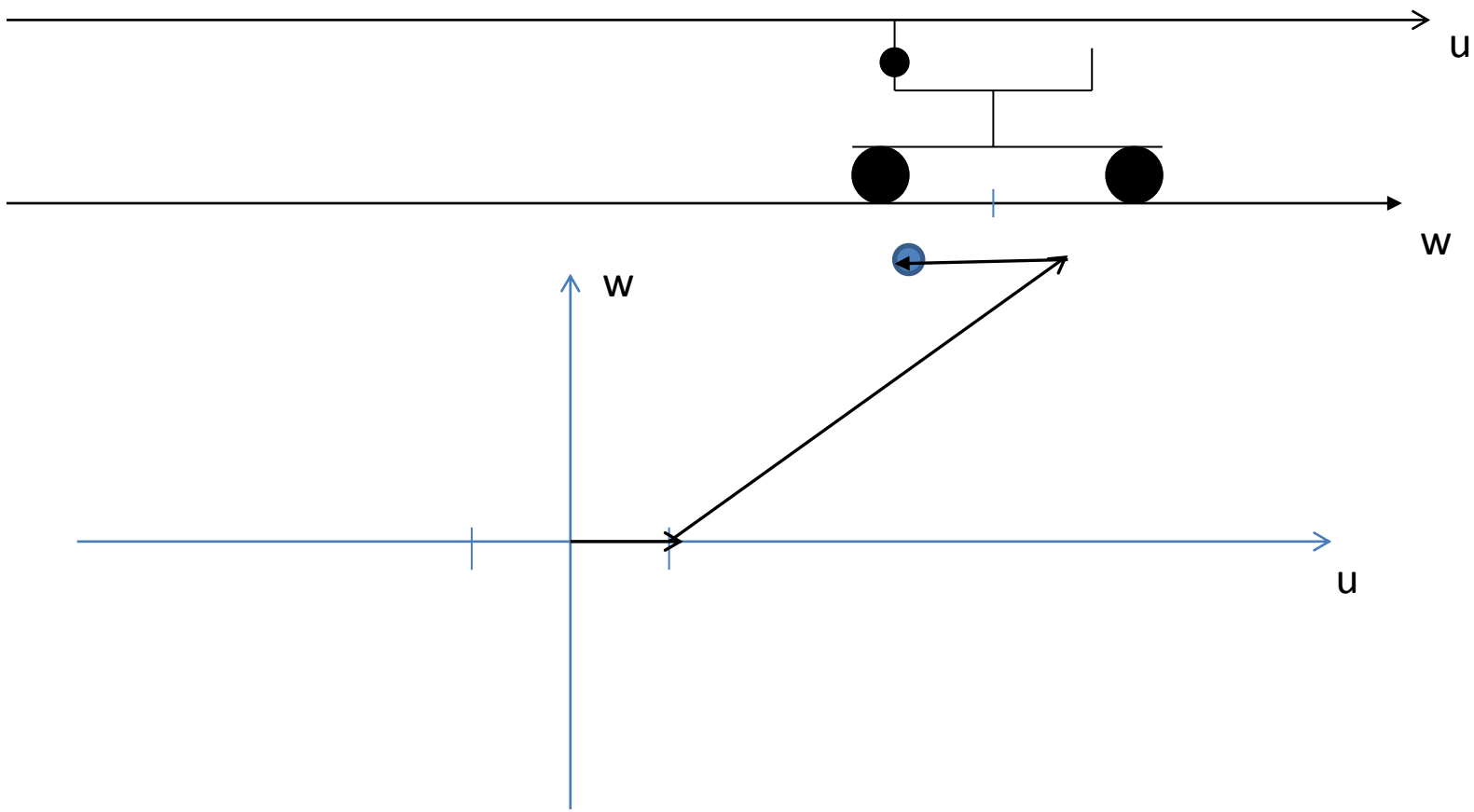

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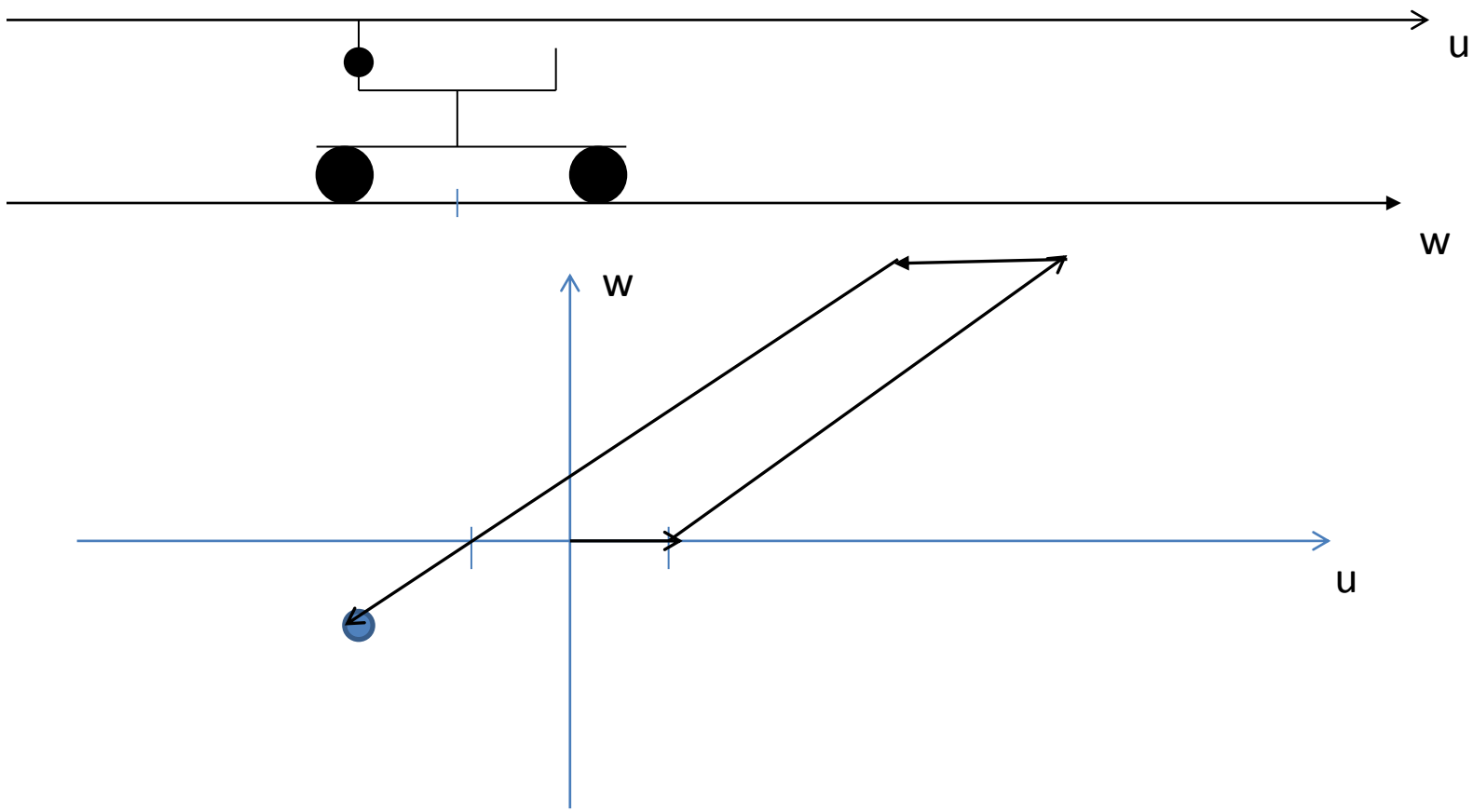
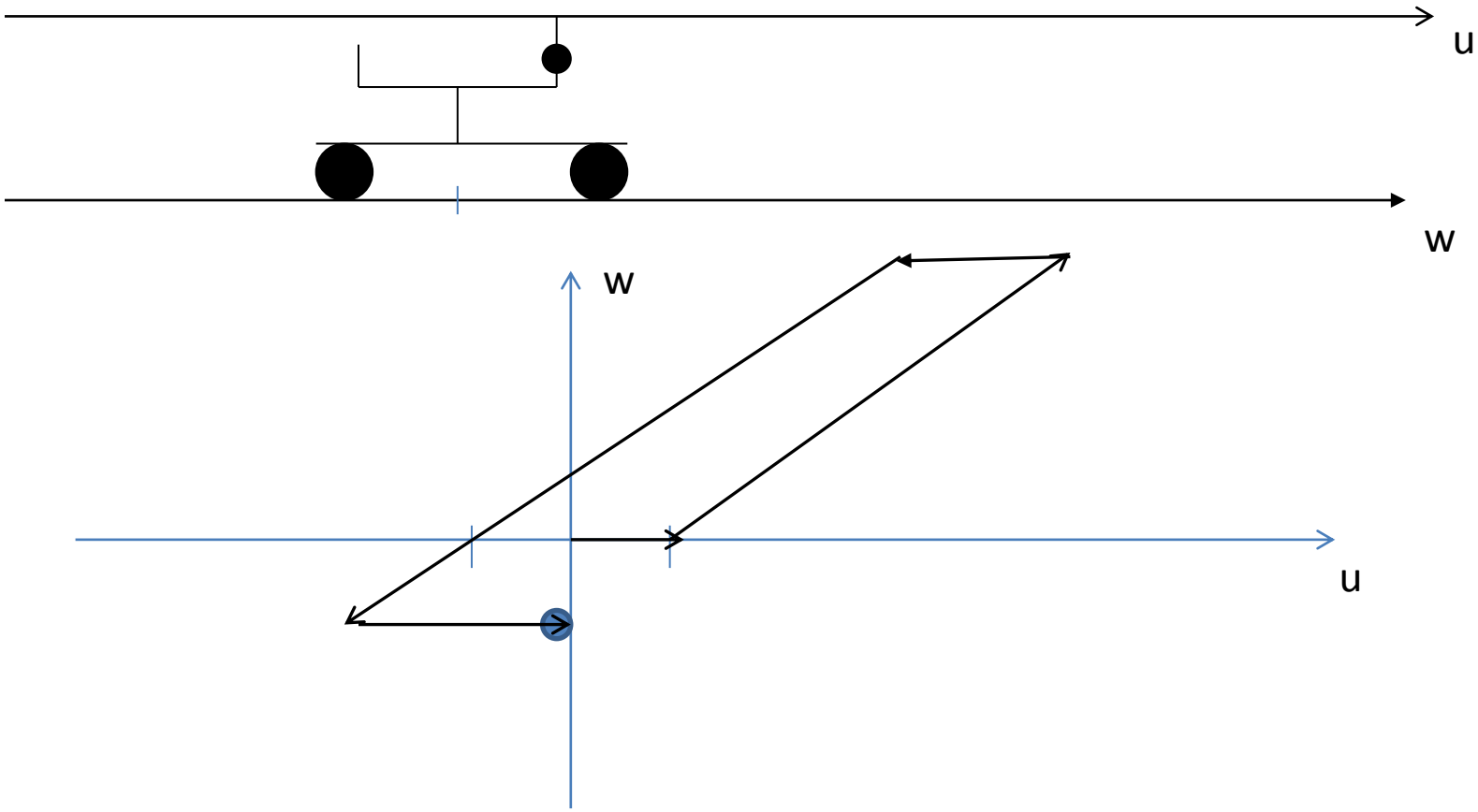
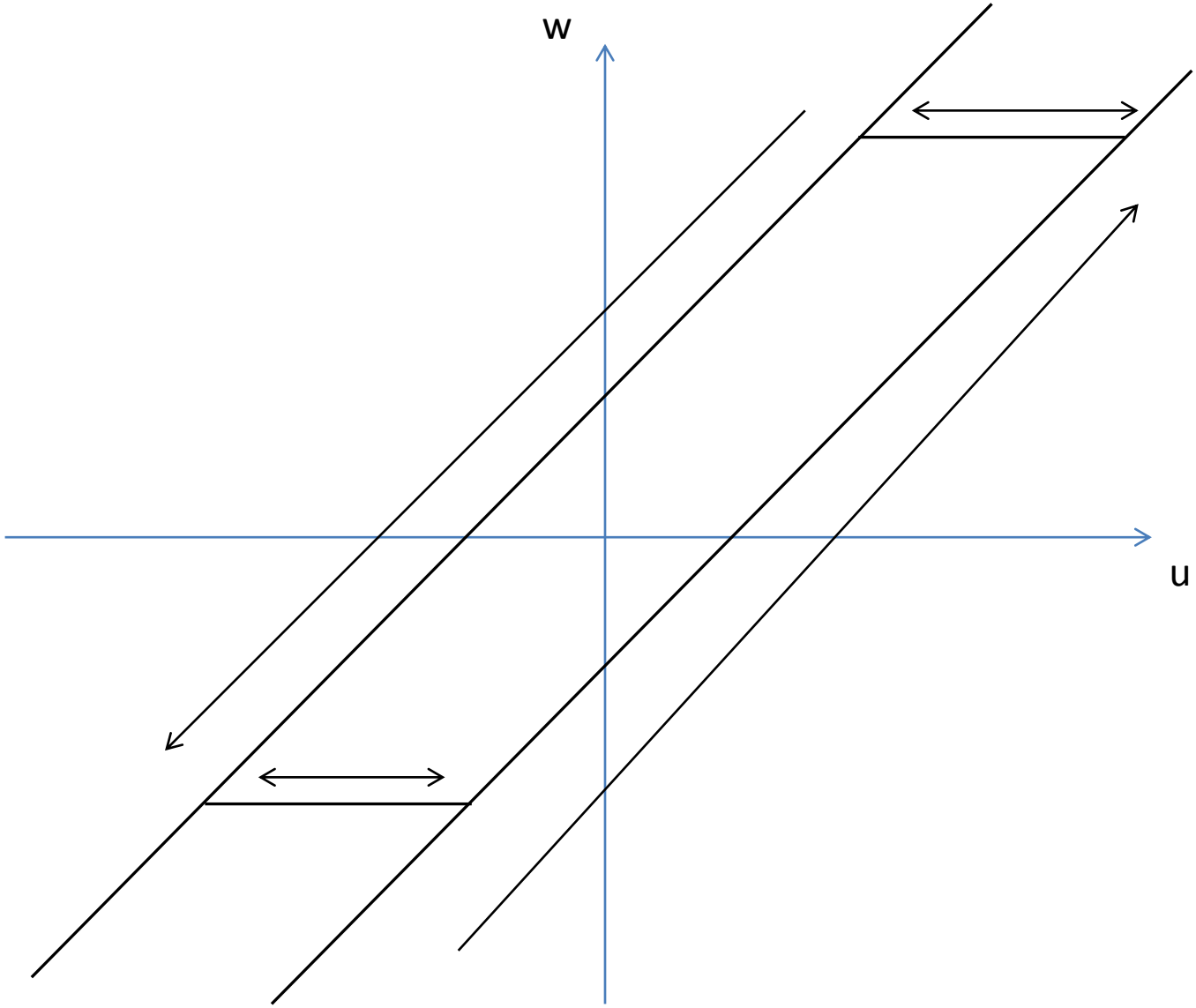


The Play operator



The Play operator



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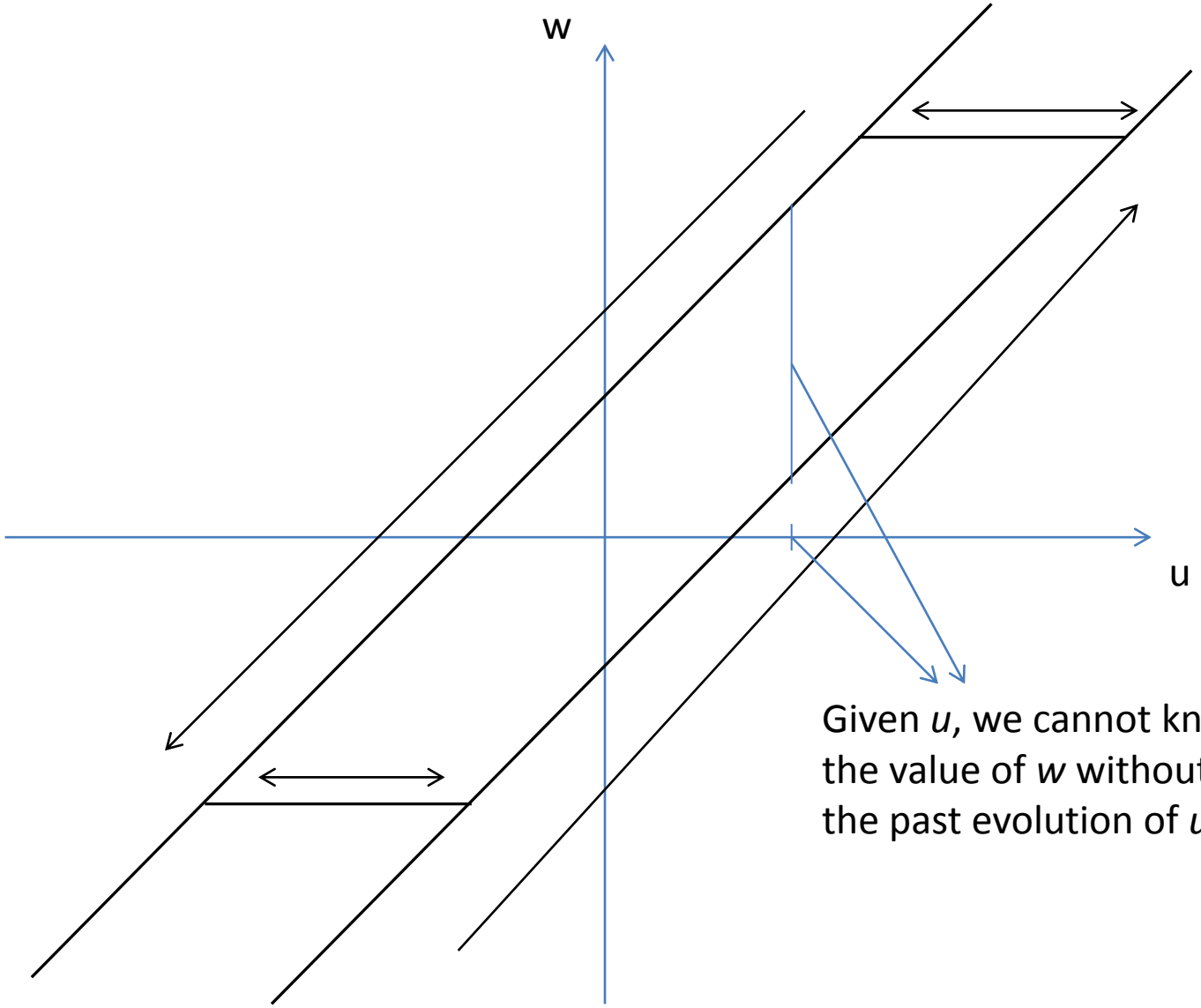
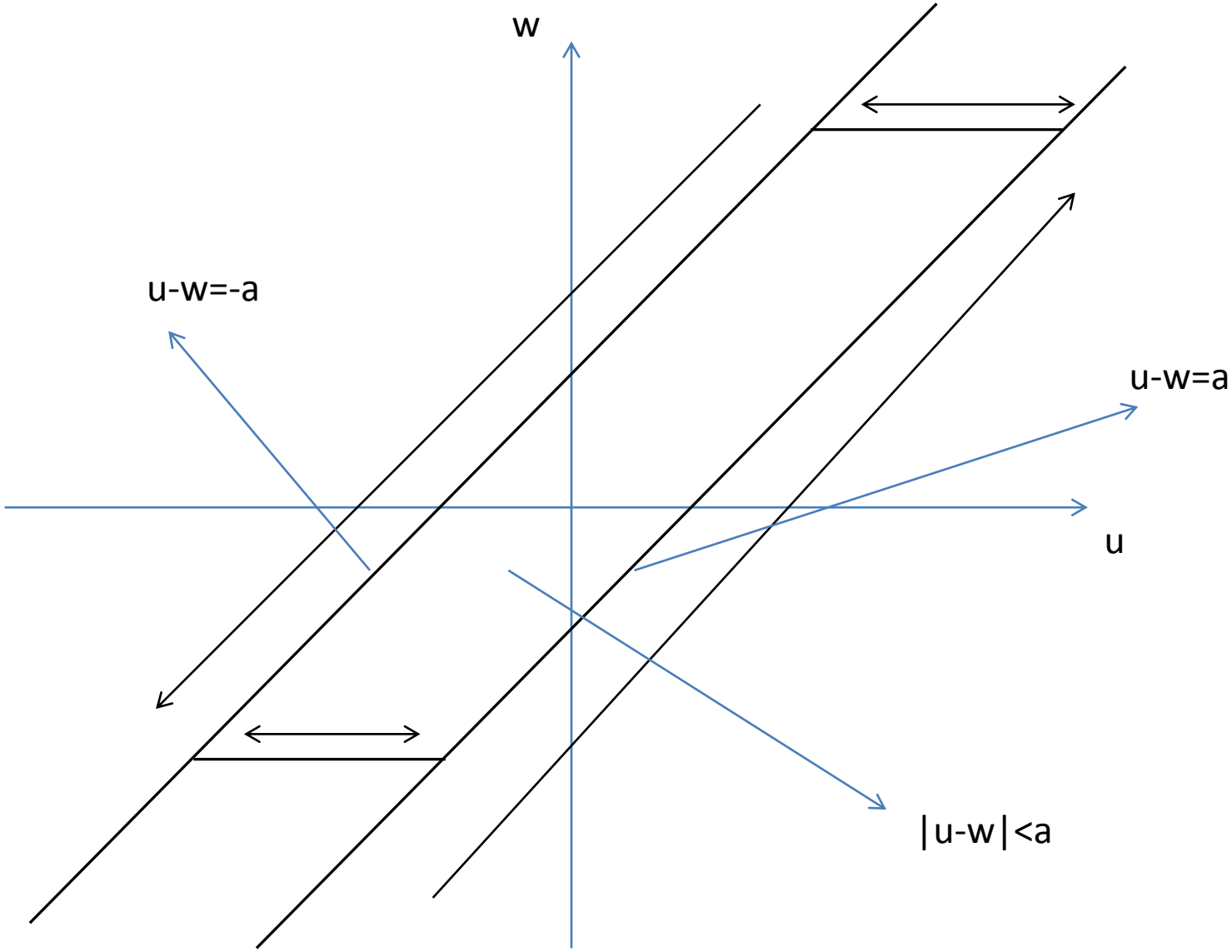


Given *u*, we cannot know the value of *w* without knowing the past evolution of *u*

The Play operator



The Play operator

Variational inequality

$$\begin{cases} w'(t)\big(u(t)-w(t)-v\big)\geq 0 \quad \forall\,|\,v\,|\leq a \quad \text{for almost every } t \\ |\,u(t)-w(t)\,|\leq a \quad \forall t \end{cases}$$

Discontinuous ODE

$$w'(t)=\chi_{\{u-w=a\}}(t)\big(u'(t)\big)^{+}-\chi_{\{u-w=-a\}}(t)\big(u'(t)\big)^{-} \quad \text{for almost every } t$$

$$\chi_{\{u-w=a\}}(t)=\begin{cases} 1 & \text{if } \; u(t)-w(t)=a \\ 0 & \text{otherwise} \end{cases}$$

$$\big(u'(t)\big)^{+}=\max\{u'(t),0\}$$

$$\big(u'(t)\big)^{-}=\max\{-u'(t),0\}$$

Hybrid Systems formulation

- ..\Hybrid\hybrid_short.pdf

ODE with hysteresis

$$\begin{cases} y'(t) = f(y(t), w(t)), \ t > 0, \\ w(t) = \mathcal{F}[y(\cdot)](t), \ t > 0, \\ y(0) = x \\ w(0) = w^0 \end{cases}$$

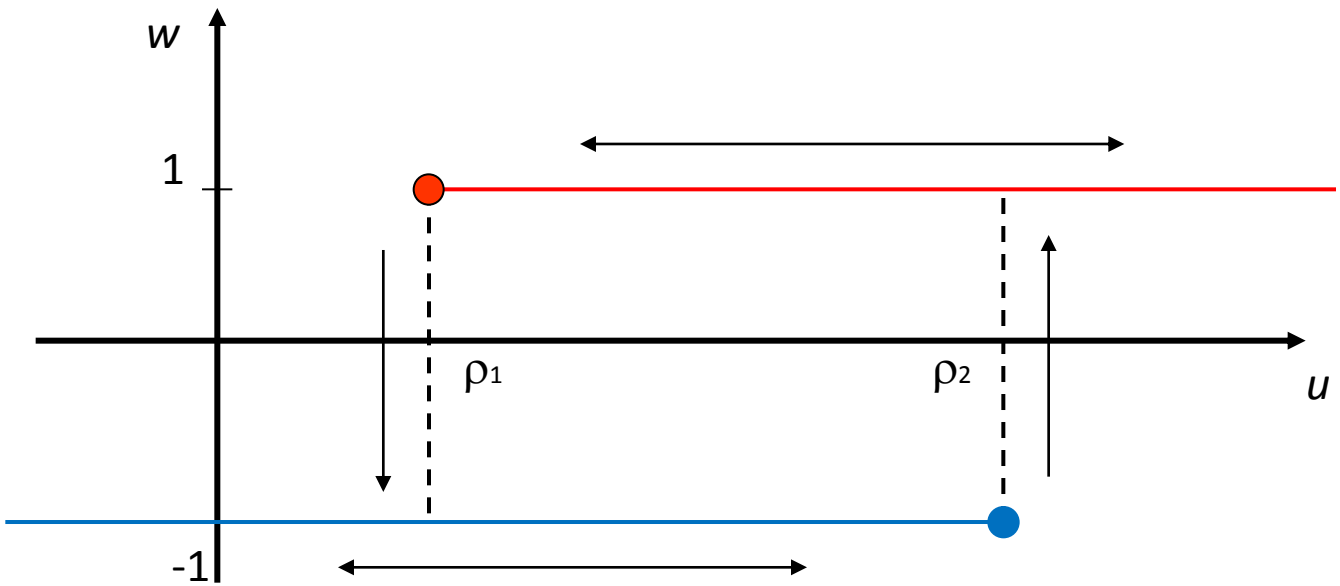
Play as hybrid system

$$\begin{cases} y' = f(y,w) \\ w = \mathcal{F}[y] \end{cases}$$

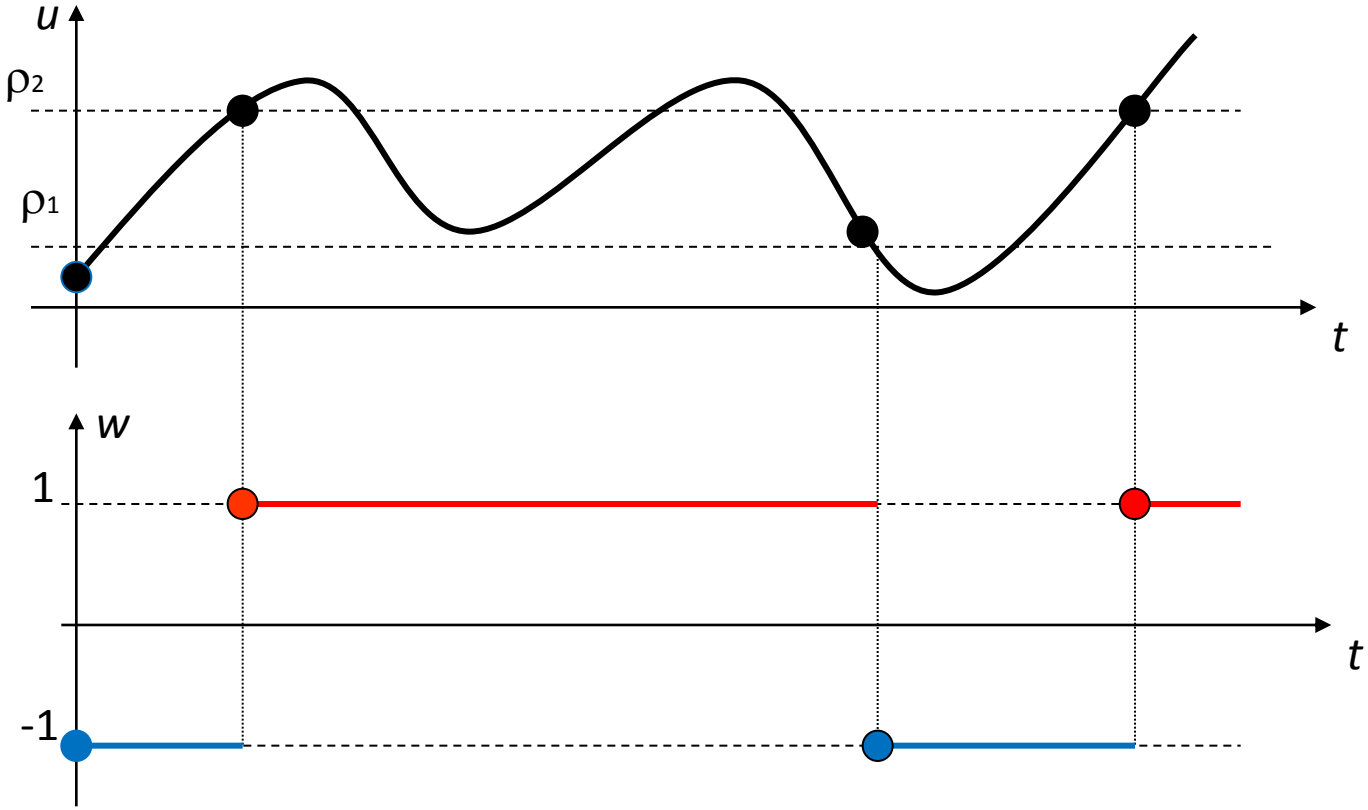
$\mathcal{F}$ is the Play operator

$$D = \{(y,w,z) \text{ s.t.} \mid y - w \mid \leq a, z = 0\},$$
$$C_1 = \{(y,w,z) \text{ s.t.} \, y - w \geq a, z \in \{0,1\}\},$$
$$C_{-1} = \{(y,w,z) \text{ s.t.} \, y - w \leq -a, z \in \{0,-1\}\}$$
$$\begin{cases} (y,w,z)' = \left(f(y,w), z \mid f(y,w) \mid, 0\right) & \text{if} \quad (y,w,z) \in D \\ (y,w,z)^+ = (y,w,z^+) = \left(y,w,g(z)\right) & \text{if} \quad (y,w,z) \in C \end{cases}$$

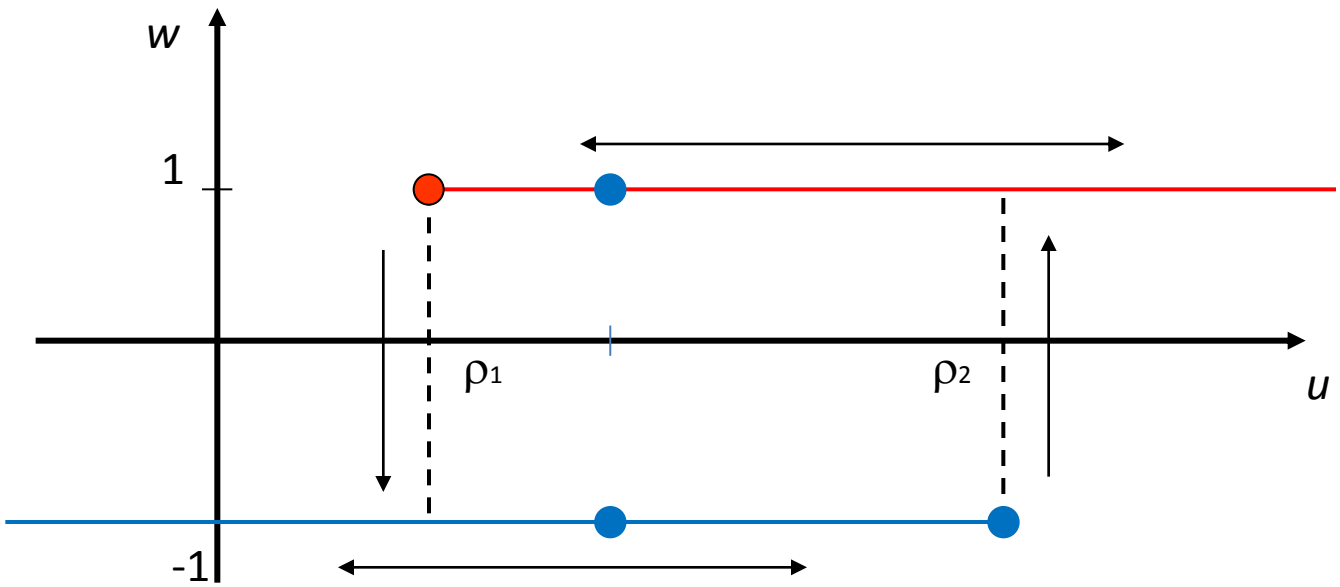
Switching hysteresis (thermostat or delayed relay)



Switching evolution of output subject to the continuous input *u*



Switching hysteresis (thermostat or delayed relay)



Delayed thermostat

$$w(t) = \begin{cases} w(0) & \text{if} \quad X_t := \{\tau \in [0,t] \text{ s.t. } u(t) = \rho_1 \text{ or u(t)} = \rho_2\} = \phi \\ -1 & \text{if} \quad u(\max X_t) = \rho_1 \\ 1 & \text{if} \quad u(\max X_t) = \rho_2 \end{cases}$$

A variational principle:

given the input u, the output w minimizes, for every t,

the total variation (number of jumps) on $[0,t]$,

among all functions ω such that

$$(u(\tau), \omega(\tau)) \in (]-\infty, \rho_2] \times \{-1\}) \cup ([\rho_1, +\infty[\times \{1\})$$

Delayed thermostat as hybrid system

$$\begin{cases} y' = f(y,w) \\ w = \mathcal{F}[y] \end{cases}$$

$\mathcal{F}$ is the delayed thermostat

$$D = \left(]-\infty, \rho_2] \times \{-1\}\right) \cup \left([\rho_1, +\infty[\times \{1\}\right)$$

$$C = \left([\rho_2, +\infty[\times \{-1\}\right) \cup \left(]-\infty, \rho_1[\times \{1\}\right)$$

$$\begin{cases} (y,w)' = \left(f(y,w), 0\right) \text{ if } (y,w) \in D \\ (y,w)^+ = (y,w^+) = \left(y, g(w)\right) \text{ if } (y,w) \in C \end{cases}$$

Optimal control problem

$$\begin{cases} y'(t) = f(y(t), w(t), \alpha(t)), \ t > 0, \\ w(t) = \mathcal{F}[y(\cdot)](t), \ t > 0, \\ y(0) = x, \quad w(0) = w^0 \end{cases}$$

$\alpha : [0, +\infty[\to A$, measurable control, $A \subset \mathbf{R}^p$ compact

Infinite horizon cost

$$J(x, w^0, \alpha) = \int_0^{+\infty} e^{-\lambda t} \ell(y(t), w(t), \alpha(t)) dt$$

Optimal control problem

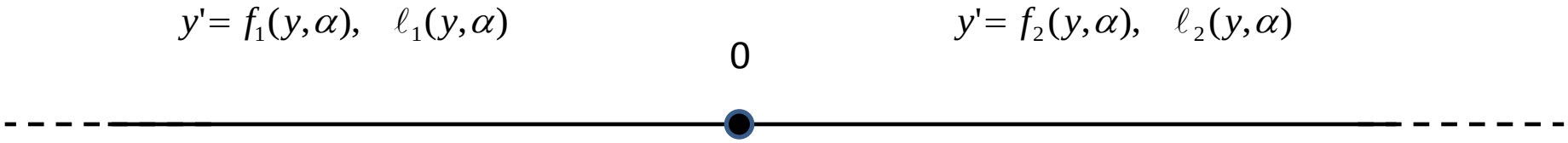
$$V(x, w^0) = \inf_{\alpha} J(x, w^0, \alpha), \text{ value function}$$

The aim is, using Dynamic Programming, to characterize the value function as the unique (viscosity) solution of a suitable “Hamilton-Jacobi-Bellman problem”.

Optimal control problem

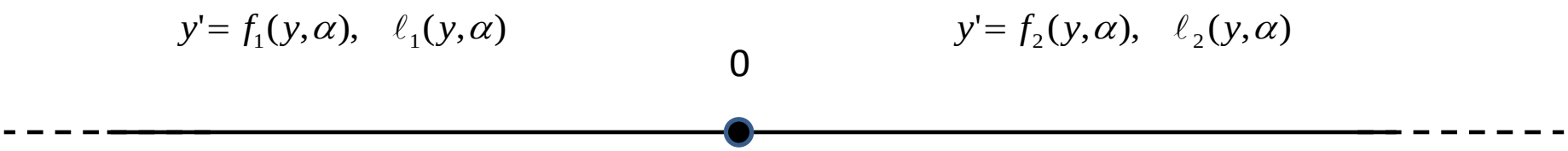
- The optimal control problem of systems intrinsically exhibiting hysteresis is a very important feature.
- Sometimes, an artificial hysteresis is introduced in the mathematical model in order to approximate some irregularities of the data, such as discontinuities and others.
- In the following two examples of “artificially introduced hysteresis” are given.

Discontinuities on the data.

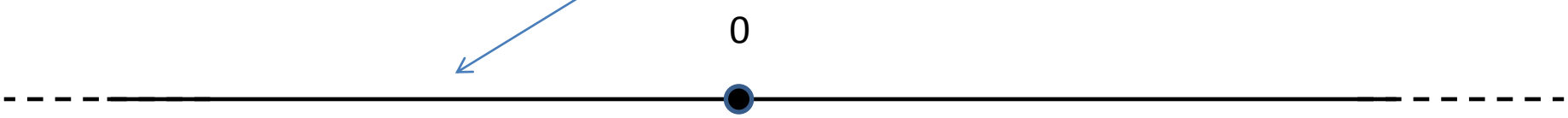


$$J(x,\alpha) = \int_0^\infty e^{-\lambda t} \ell_i\,(y_x(t,\alpha), \alpha(t))dt$$

$$\mathrm{V}(x) = \inf_{\alpha \in A} J(x,\alpha)\ \forall\, x \in R$$



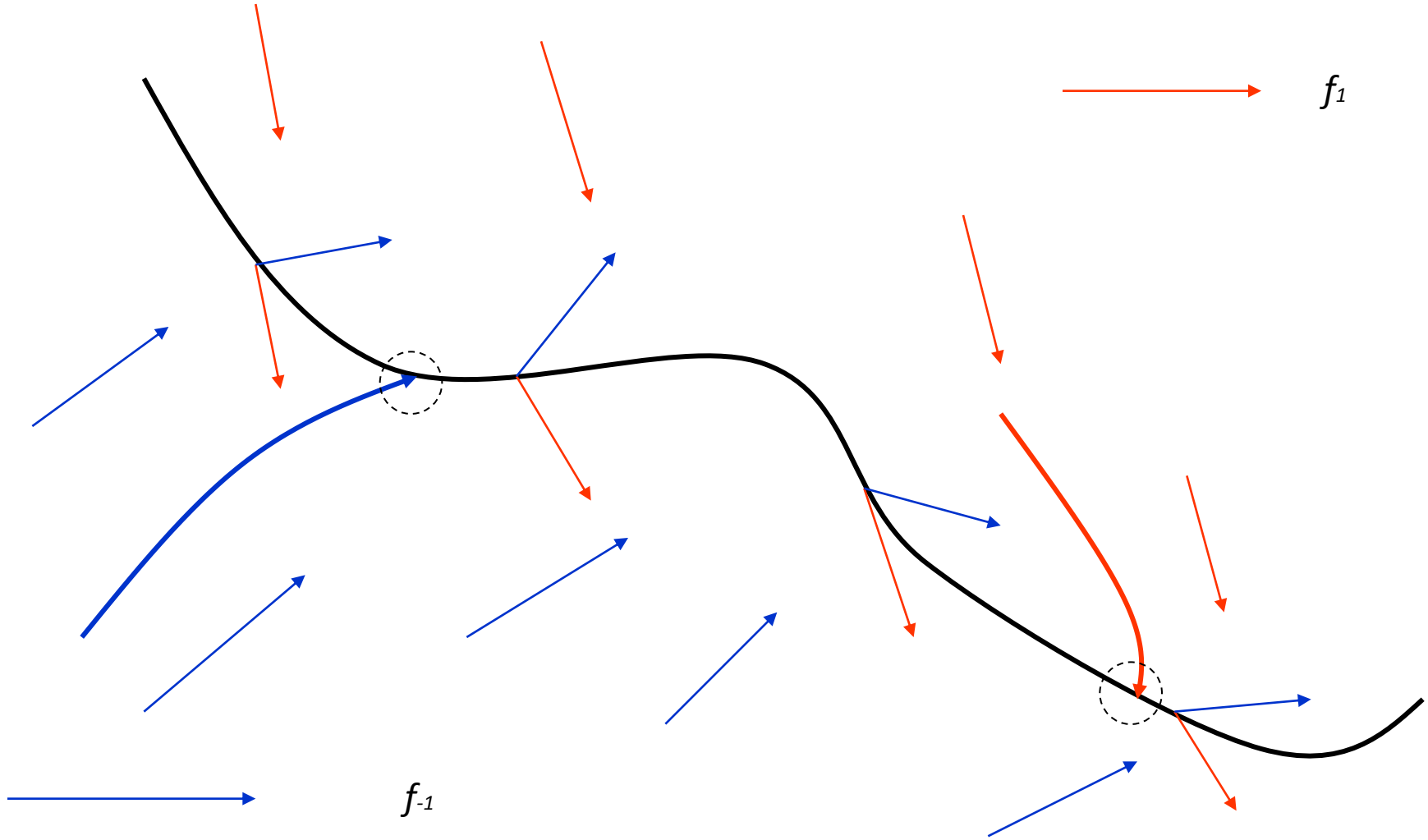
$$\lambda V(x) + H_1\big(x, \nabla V(x)\big) = \lambda V(x) + \max_a \big\{ - f_1(x,a) \cdot \nabla V(x) - \ell_1(x,a) \big\} = 0$$



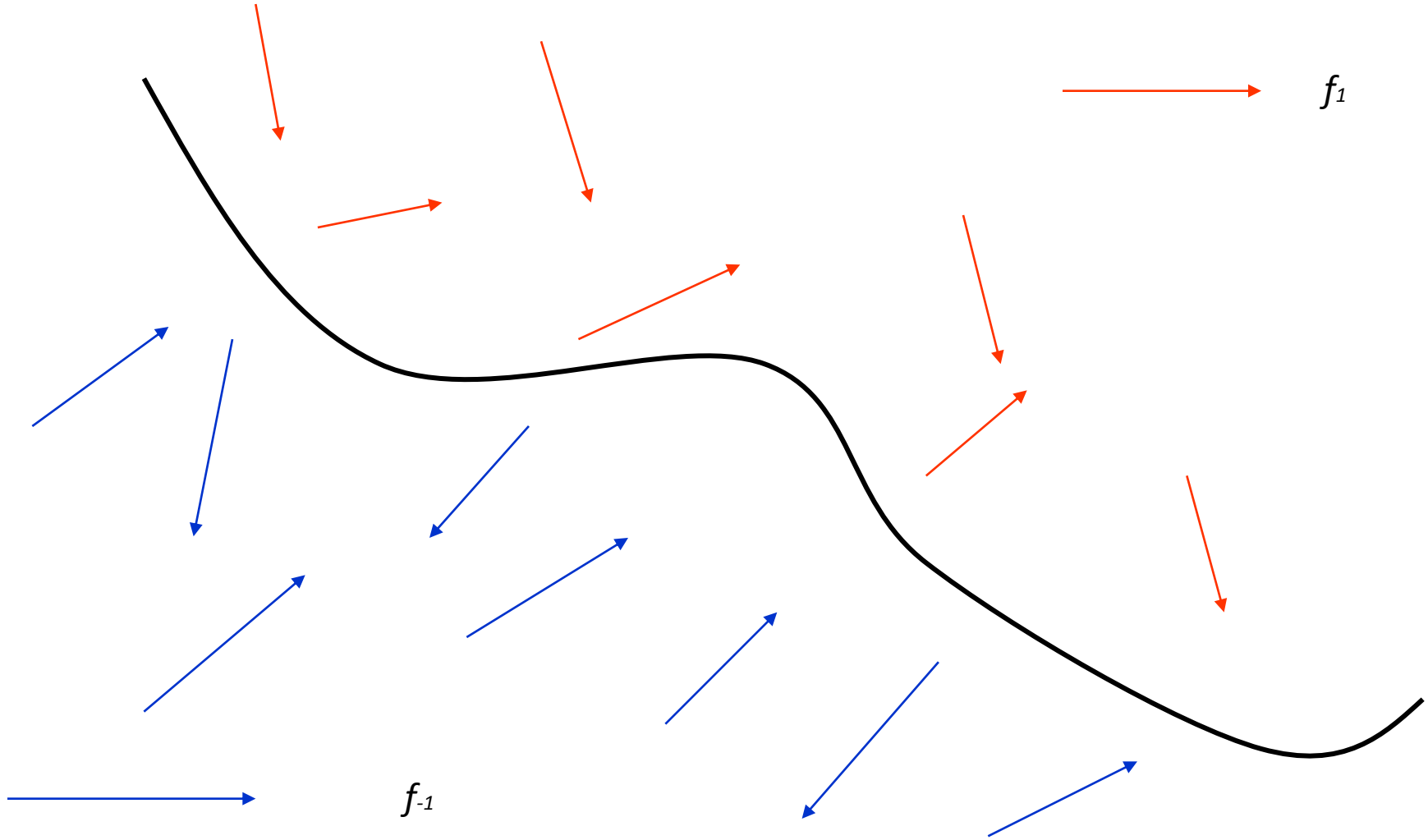
$$\lambda V(x) + H_2\big(x, \nabla V(x)\big) = \lambda V(x) + \max_a \big\{ - f_2(x,a) \cdot \nabla V(x) - \ell_2(x,a) \big\} = 0$$

Two different Hamilton-Jacobi equations on the two half lines.
Which condition on the junction x=0, in order to characterize the value function as the unique solution?

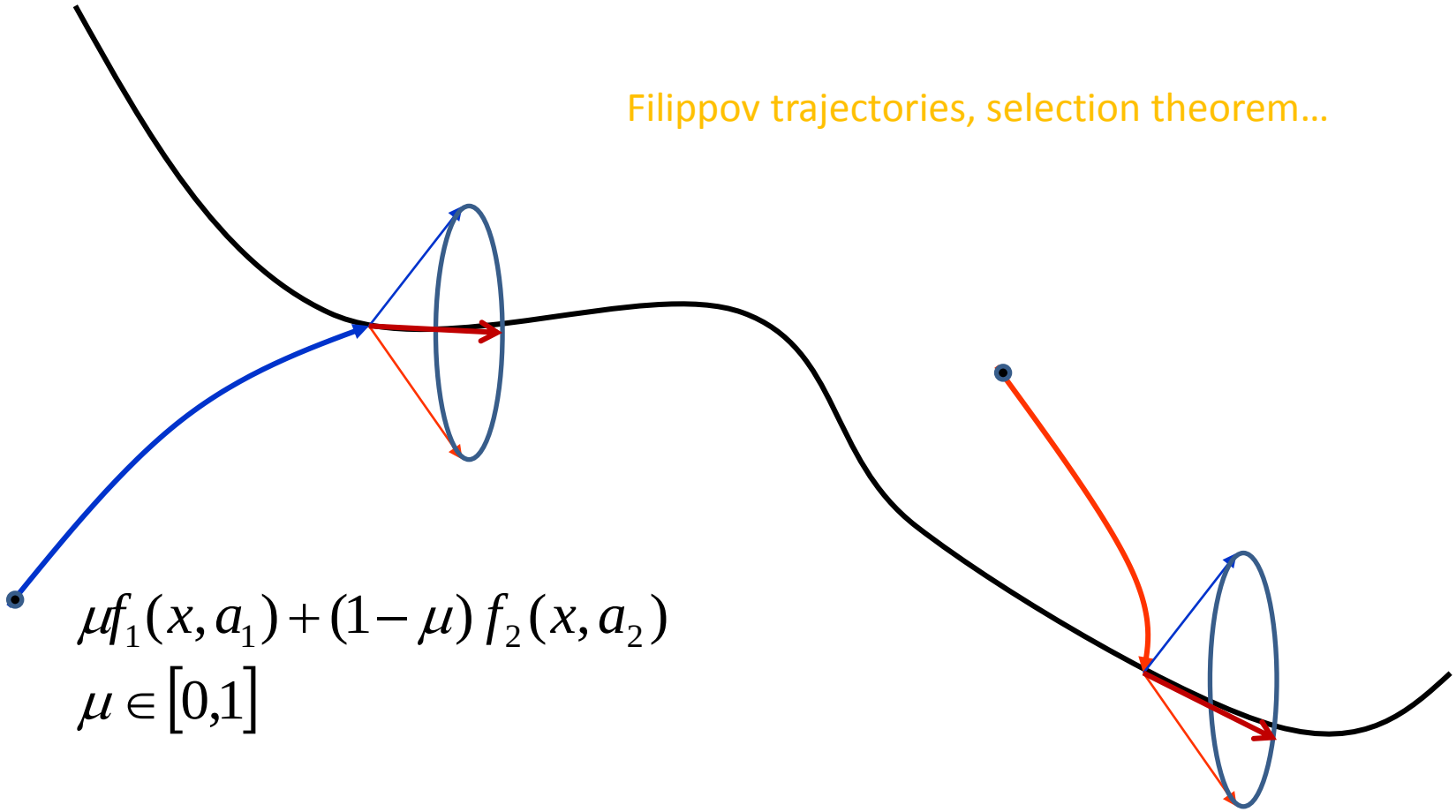
Discontinuous dynamics



Discontinuous dynamics



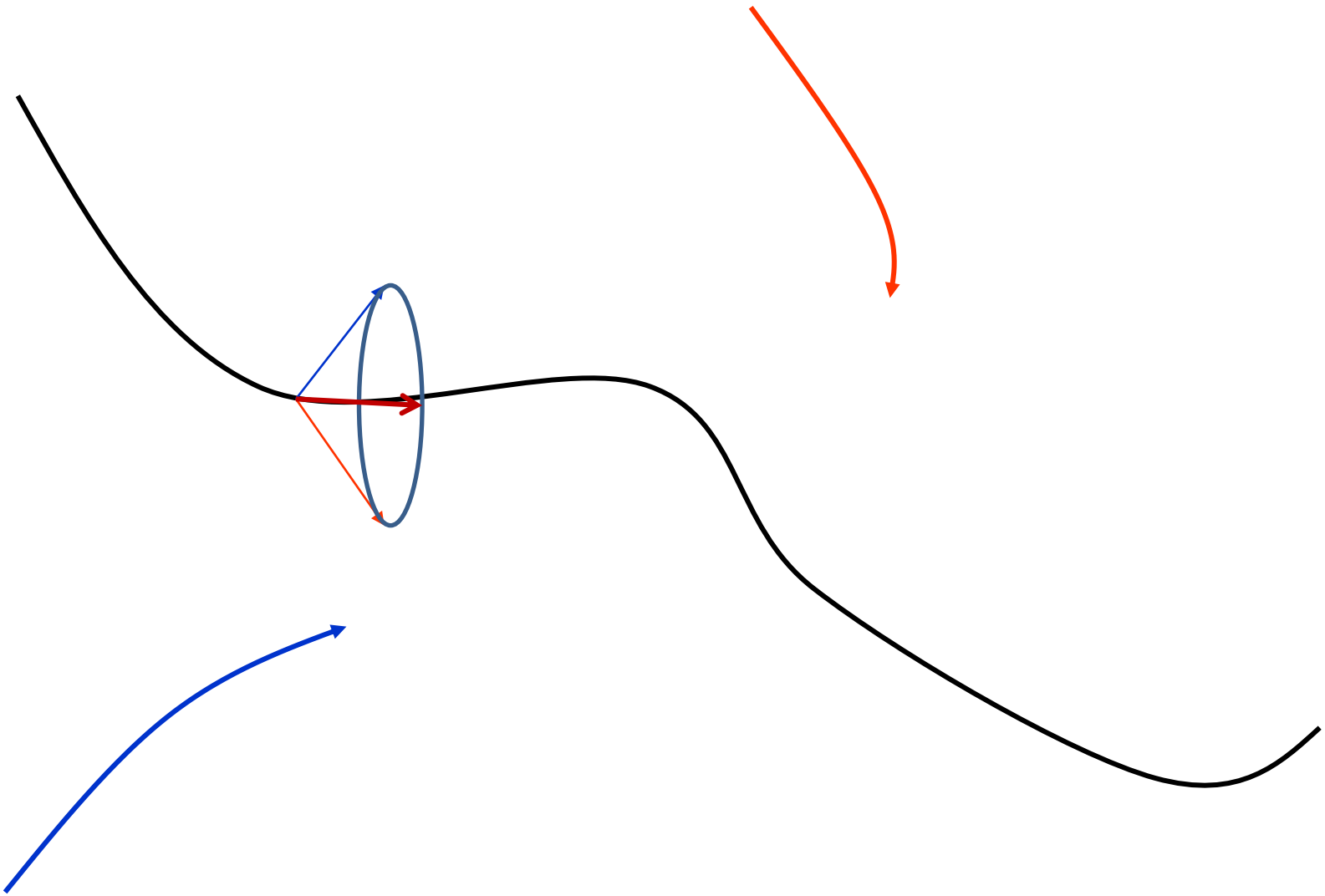
Sliding Modes



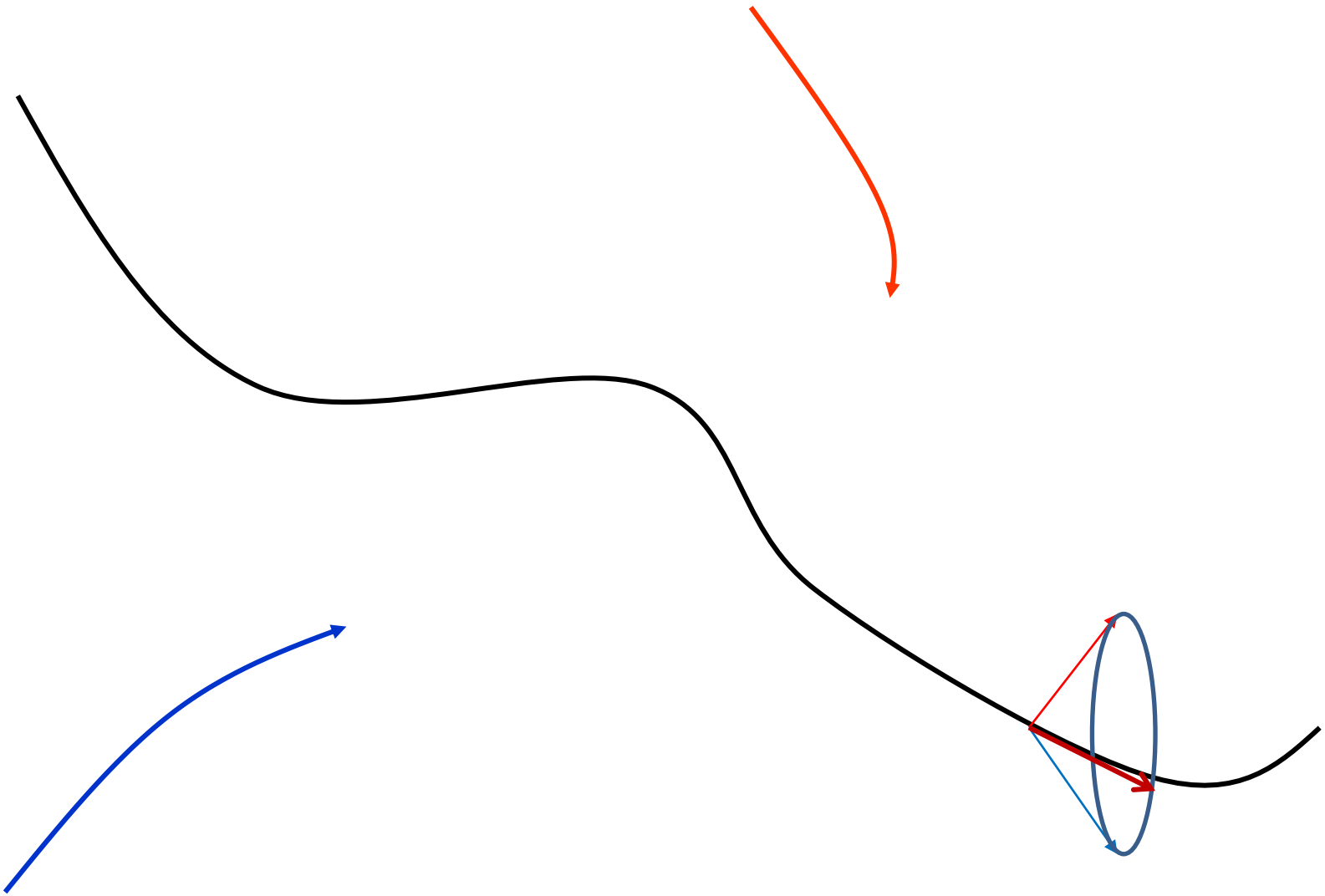
$$\mu f_1(x, a_1) + (1-\mu) f_2(x, a_2)$$

$$\mu \in [0,1]$$

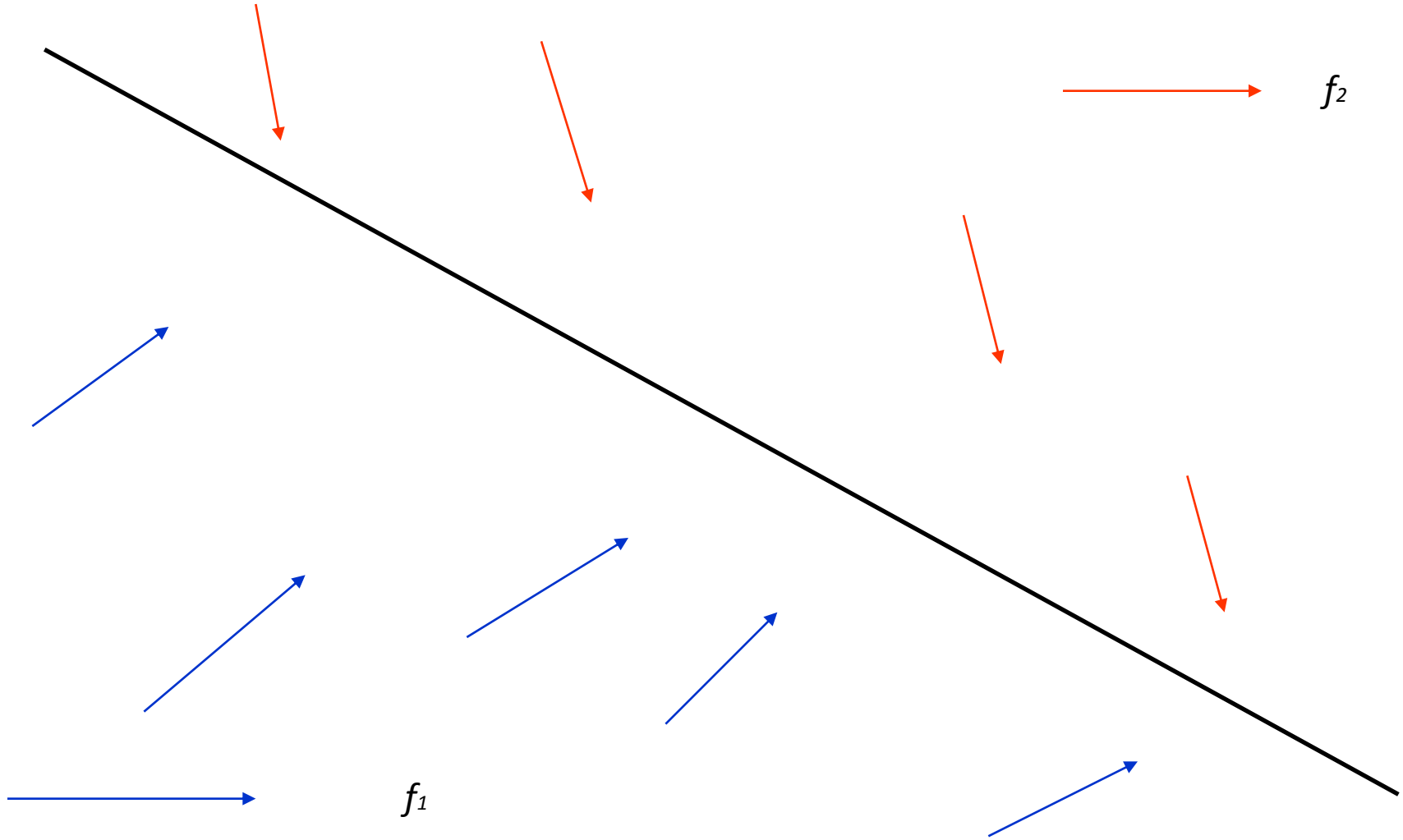
Regular tangential dynamics



Singular tangential dynamics



Discontinuous dynamics



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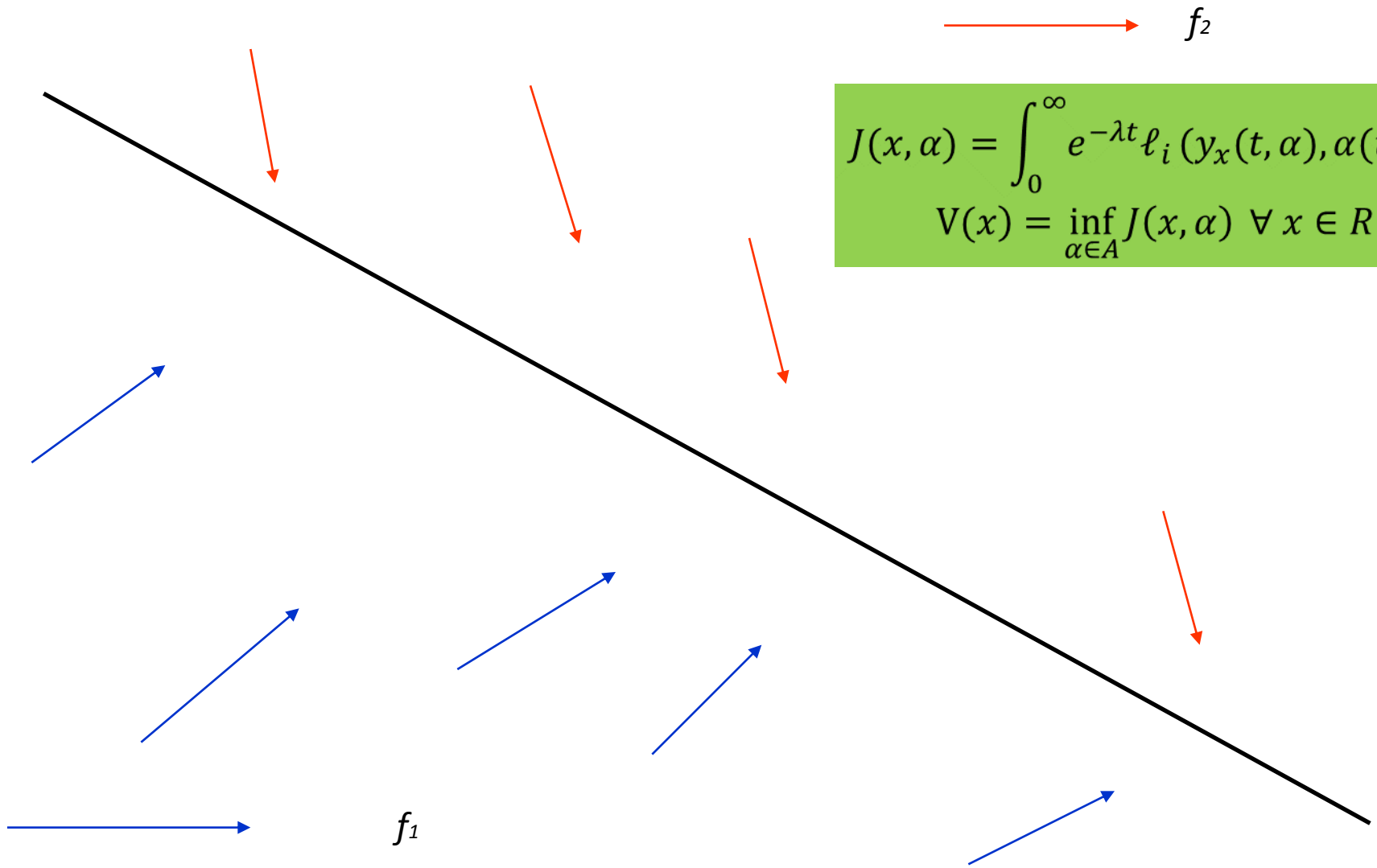


$$J(x,\alpha) = \int_0^\infty e^{-\lambda t} \ell_i\,(y_x(t,\alpha),\alpha(t))dt$$

$$\mathrm{V}(x) = \inf_{\alpha \in A} J(x,\alpha)\ \forall\, x \in R$$

Hamilton-Jacobi on multi-domains

$$\lambda V(x) + \max_{a \in A} \left\{ - f_2(x,a) \cdot \nabla V(x) - \ell_2(x,a) \right\} = \lambda V(x) + H_2(x, \nabla V(x)) = 0$$

Problem :

determine a suitable Hamilton - Jacobi problem

on $\mathbf{R}^n$ as well as "restricted to Γ"

such that it has a unique solution V on $\mathbf{R}^n$

and to give a meaning to V

as value function of suitable optimal control problem

Γ

$$\lambda V(x) + \max_{a \in A} \left\{ - f_1(x,a) \cdot \nabla V(x) - \ell_1(x,a) \right\} = \lambda V(x) + H_1(x, \nabla V(x)) = 0$$

One recent result in literature

Under suitable controllability hypotheses,

the maximal subsolution U^+ of the following problem

$$\begin{cases} \lambda V + H_2(x, \nabla V) = 0 \text{ in } \left(\mathbf{R}^n \setminus \Gamma\right)_2 \\ \lambda V + H_\Gamma^{reg}(x, \nabla V) = 0 \text{ on } \Gamma \\ \lambda V + H_1(x, \nabla V) = 0 \text{ in } \left(\mathbf{R}^n \setminus \Gamma\right)_1 \\ \max\left\{\lambda V + H_1, \lambda V + H_2\right\} \geq 0 \quad \text{on } \Gamma \\ \min\left\{\lambda V + H_1, \lambda V + H_2\right\} \leq 0 \quad \text{on } \Gamma \end{cases}$$

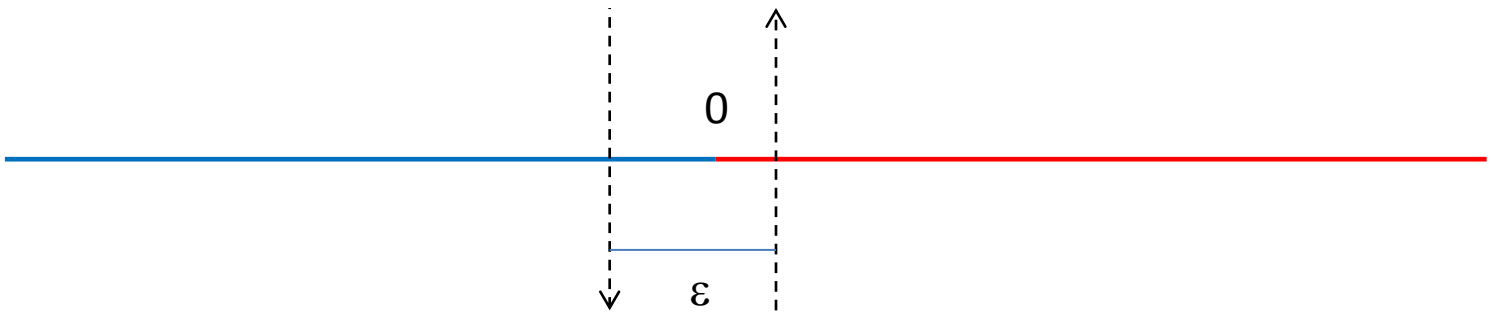
where H_Γ^{reg} is the tangential Hamiltonian on Γ taking into account

only the regular tangential trajectories,

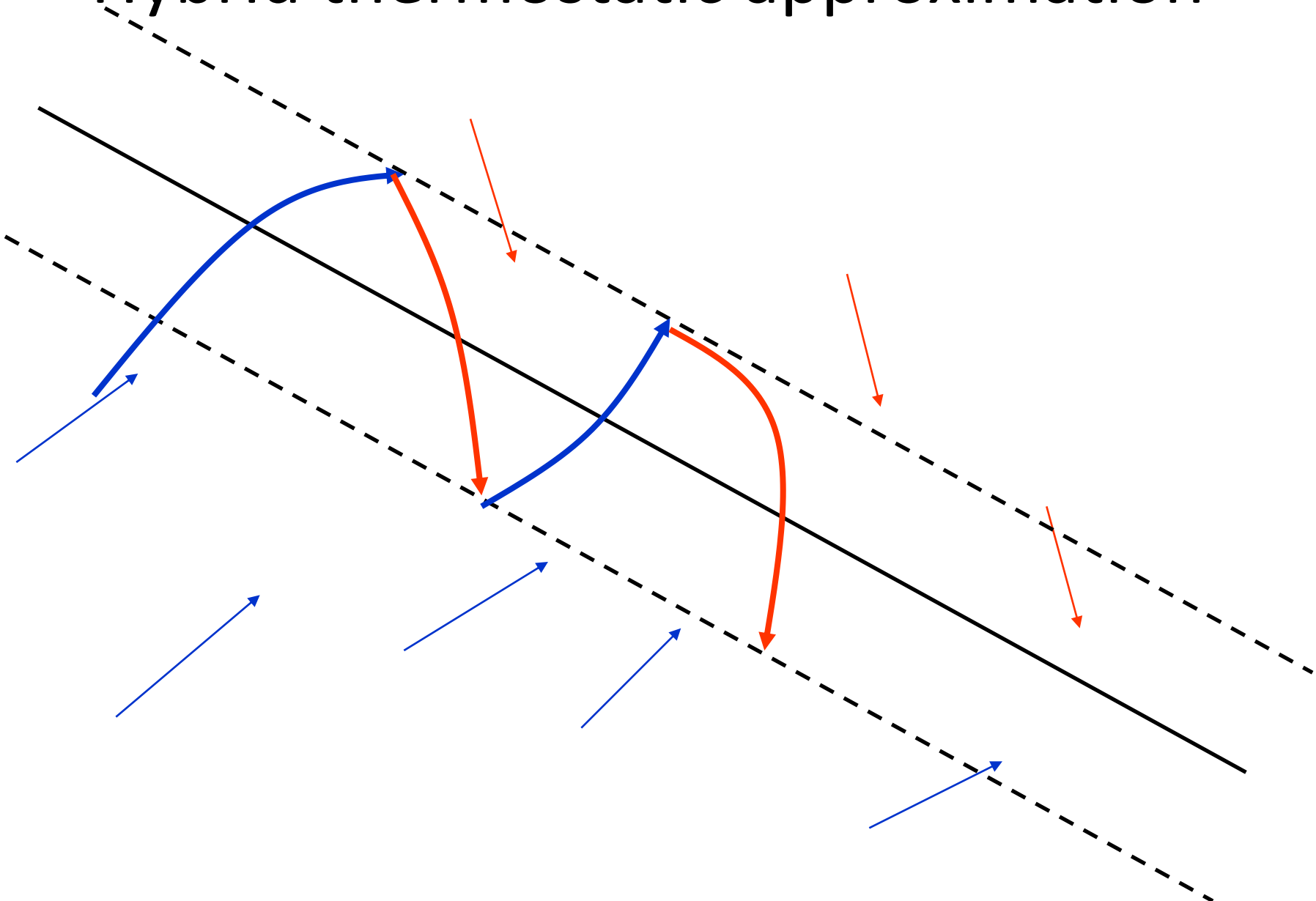
is the value function of the optimal control problem

with only regular tangential dynamics.

An hybrid thermostatic approximation of the discontinuity



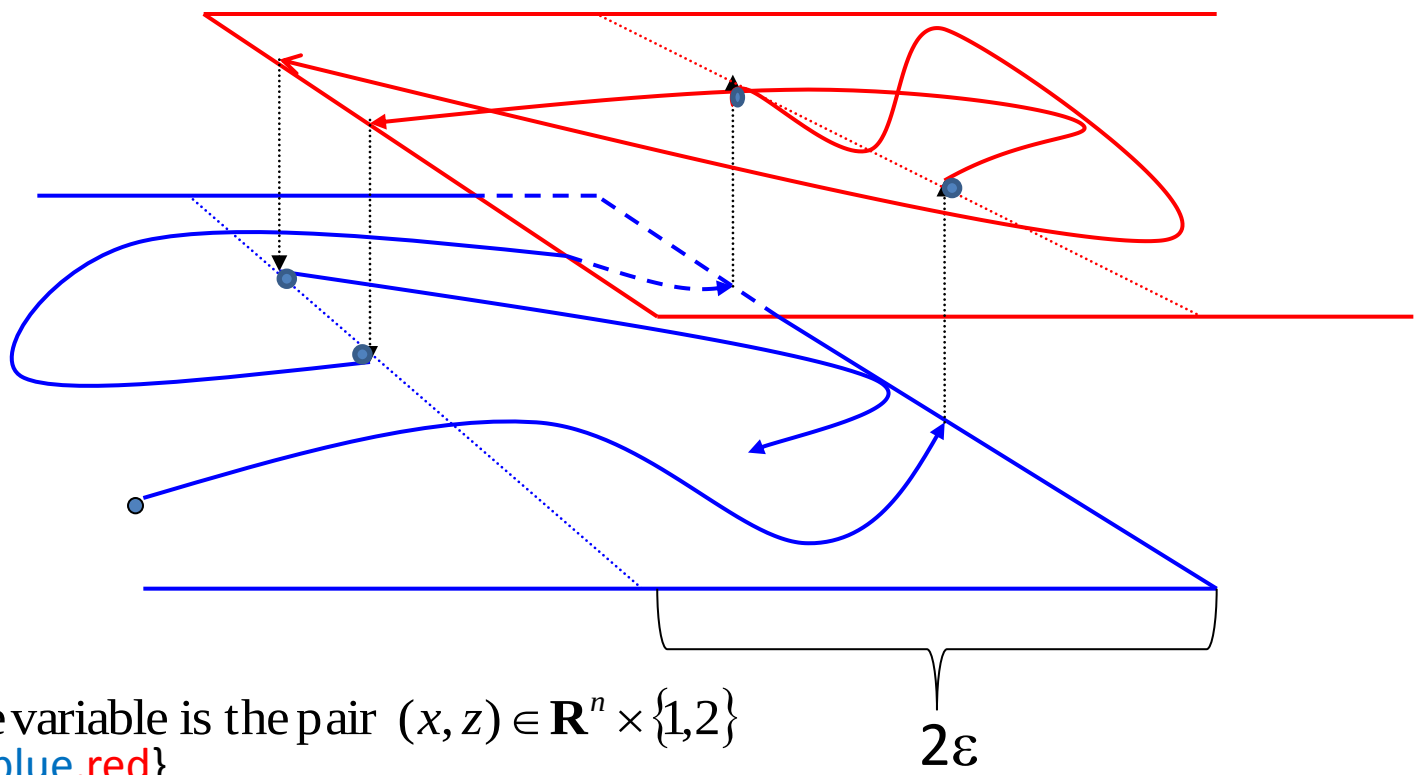
Hybrid thermostatic approximation



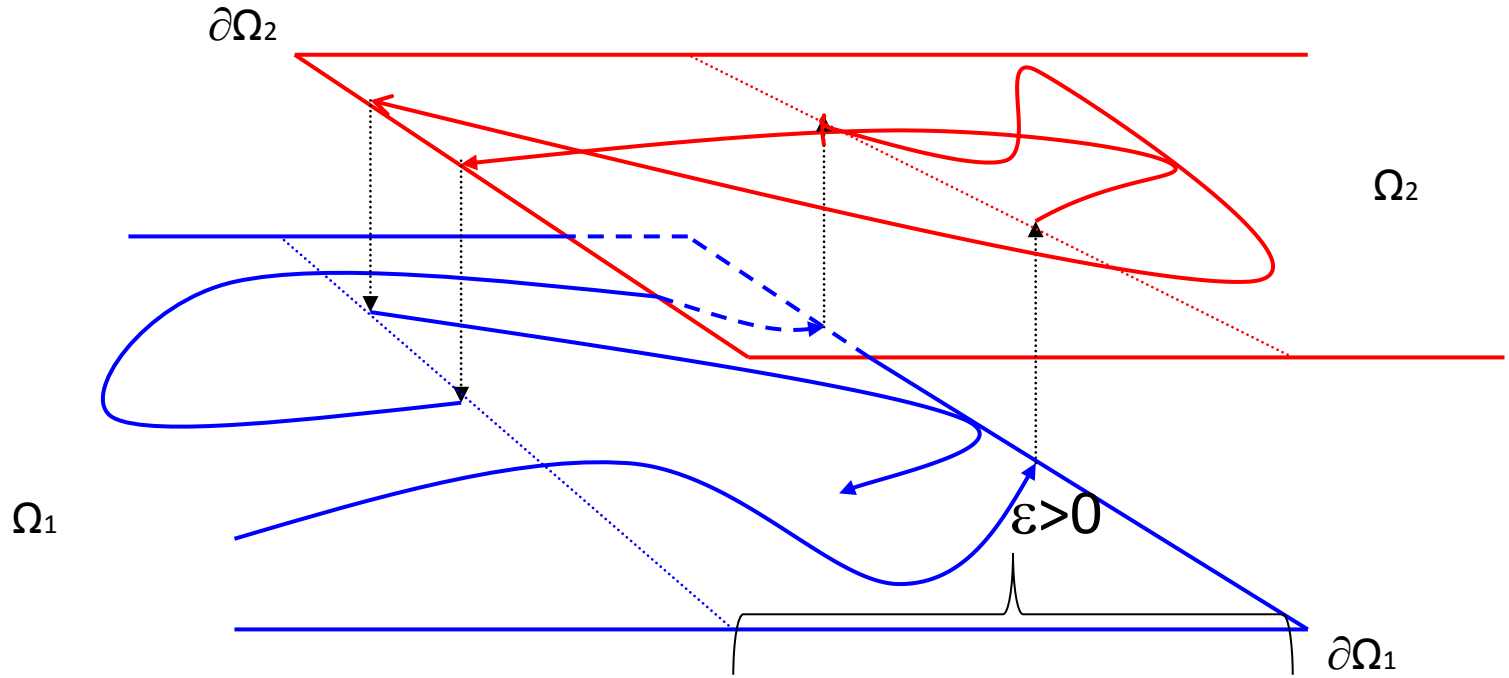
The evolution of *(y,z)*

governed by $f_2(\cdot,\alpha)$

governed by $f_1(\cdot,\alpha)$



The new state variable is the pair $(x,z) \in \mathbf{R}^n \times \{1,2\}$
$=\mathbf{R}n \times \{\text{blue},\text{red}\}$

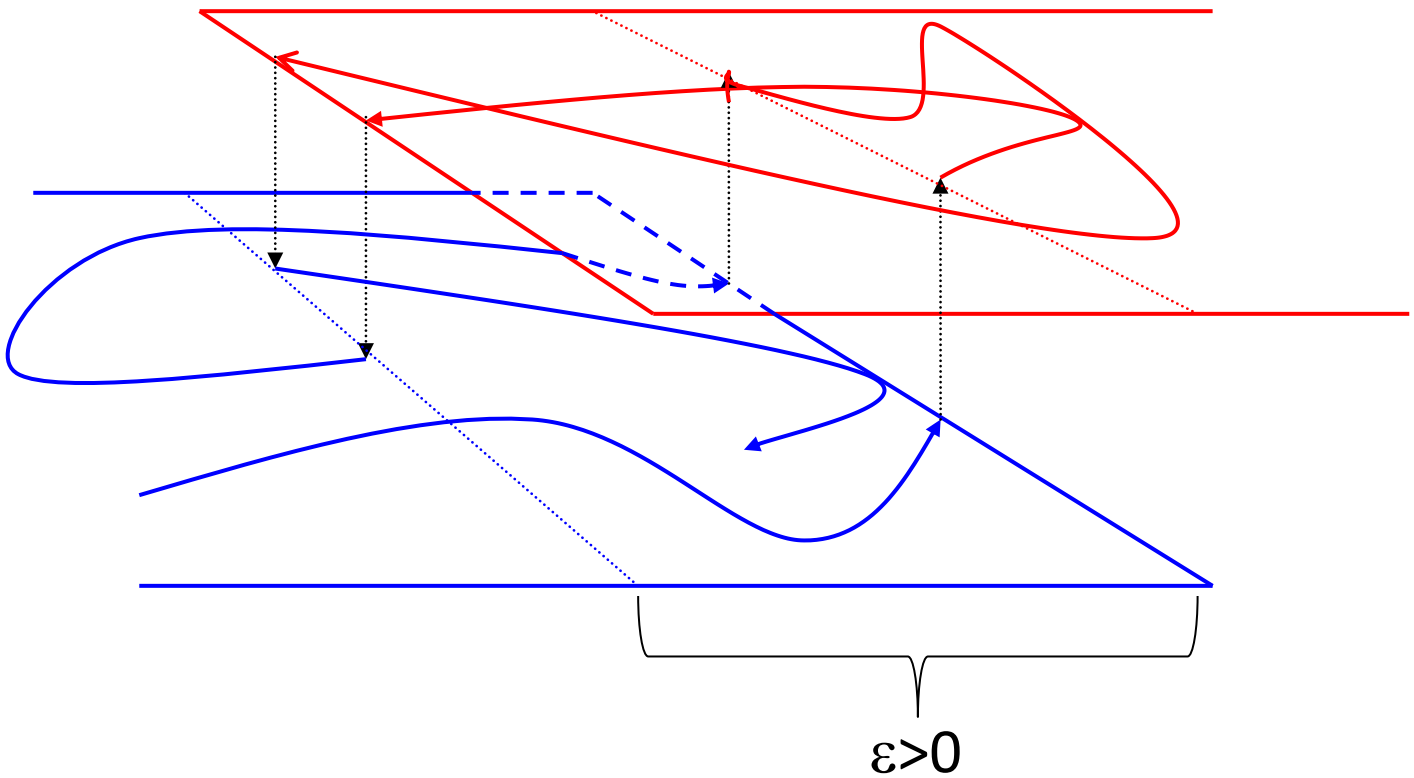


We can think to our problem as a coupling of two problems of exit-time type which mutually exchange their exit-costs (boundary conditions)

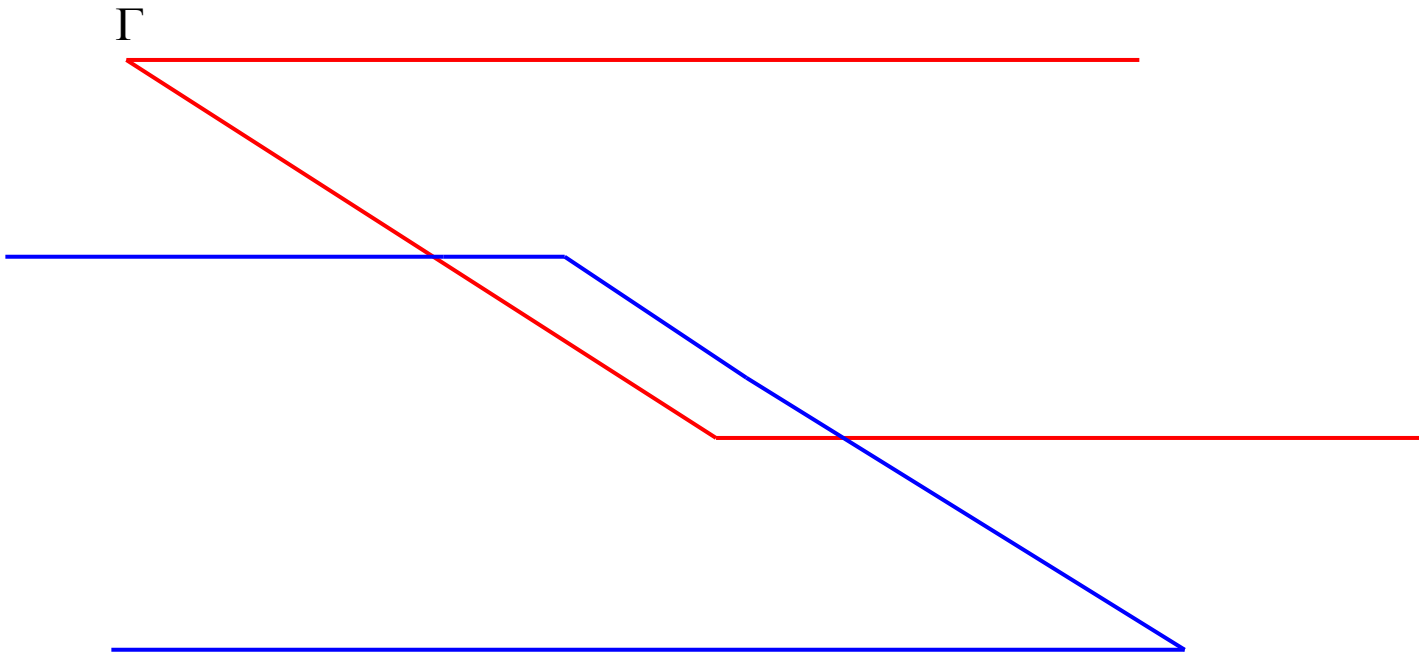
$$\wp_\varepsilon \begin{cases} \begin{cases} \lambda V_\varepsilon(x,1) + \max\limits_{a\in A}\{-f_1(x,a)\cdot\nabla V_\varepsilon(x,1) - \ell_1(x,a)\} = 0 \quad \text{in } \Omega_1^\varepsilon \\ V_\varepsilon(x,1) = V_\varepsilon(x,2) \quad \text{on } \partial\Omega_1^\varepsilon \end{cases} \\ \begin{cases} \lambda V_\varepsilon(x,2) + \max\limits_{a\in A}\{-f_2(x,a)\cdot\nabla V_\varepsilon(x,-1) - \ell_2(x,a)\} = 0 \quad \text{in } \Omega_2^\varepsilon \\ V_\varepsilon(x,2) = V_\varepsilon(x,1) \quad \text{on } \partial\Omega_2^\varepsilon \end{cases} \end{cases}$$

In series of papers it is proven that the value function of this (and others) switching thermostatic problem is the unique viscosity solution of the coupled exit-time problem.

What happens when $\varepsilon \to 0$?



Obviously, we recover a situation as the multi-domains situation.



Theorem: Let $\tilde{V}_\varepsilon : R^2 \setminus \{x = 0\} \to \mathbf{R}$ be

$$\tilde{V}_\varepsilon = \begin{cases} V_\varepsilon(x, y, 1) & \text{if} \quad x \geq 0, y \in \mathbf{R} \\ V_\varepsilon(x, y, 2) & \text{if} \quad x \leq 0, y \in \mathbf{R} \end{cases}$$

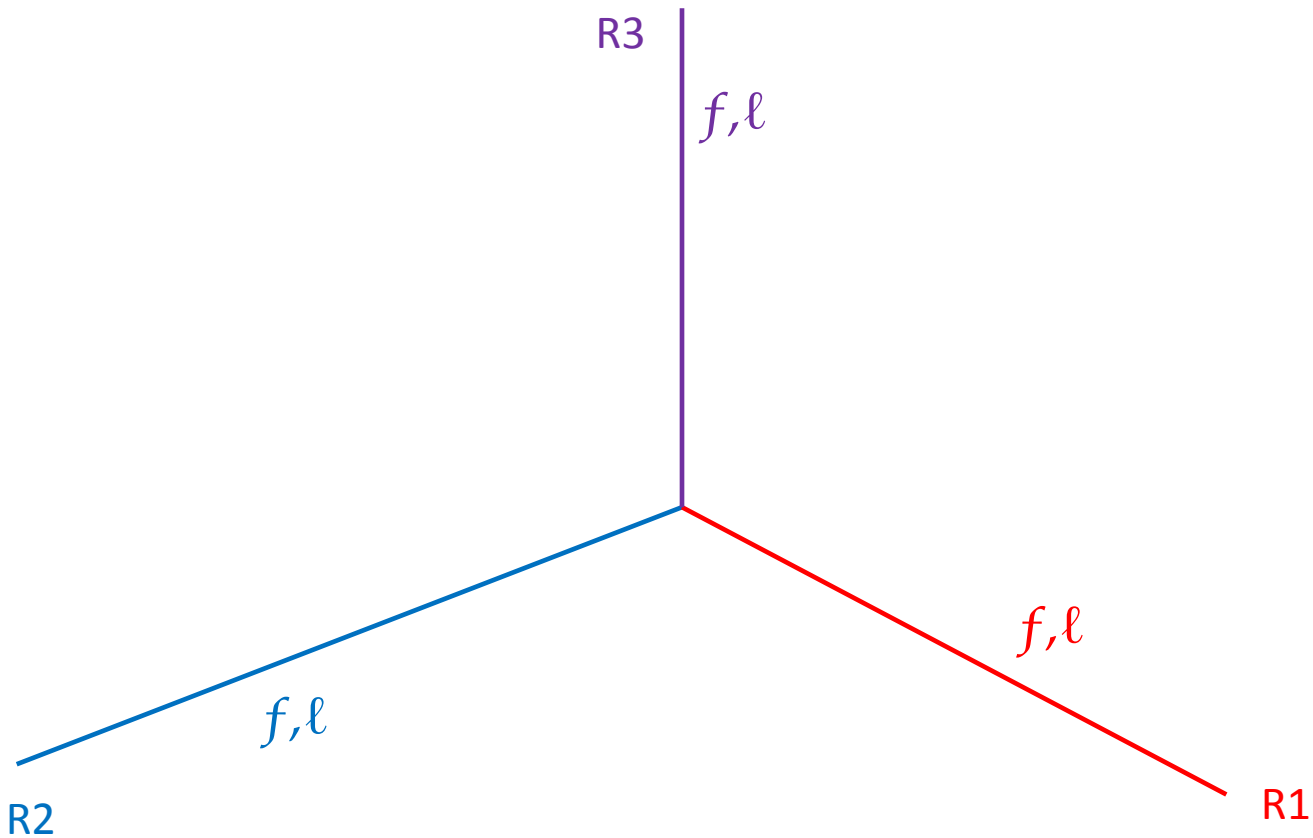
As $\varepsilon \to 0$, the sequence of function $\tilde{V}_\varepsilon$ uniformly converges on $\mathbf{R}^2 \setminus \{x = 0\}$ to a continuous function $\tilde{V}$.

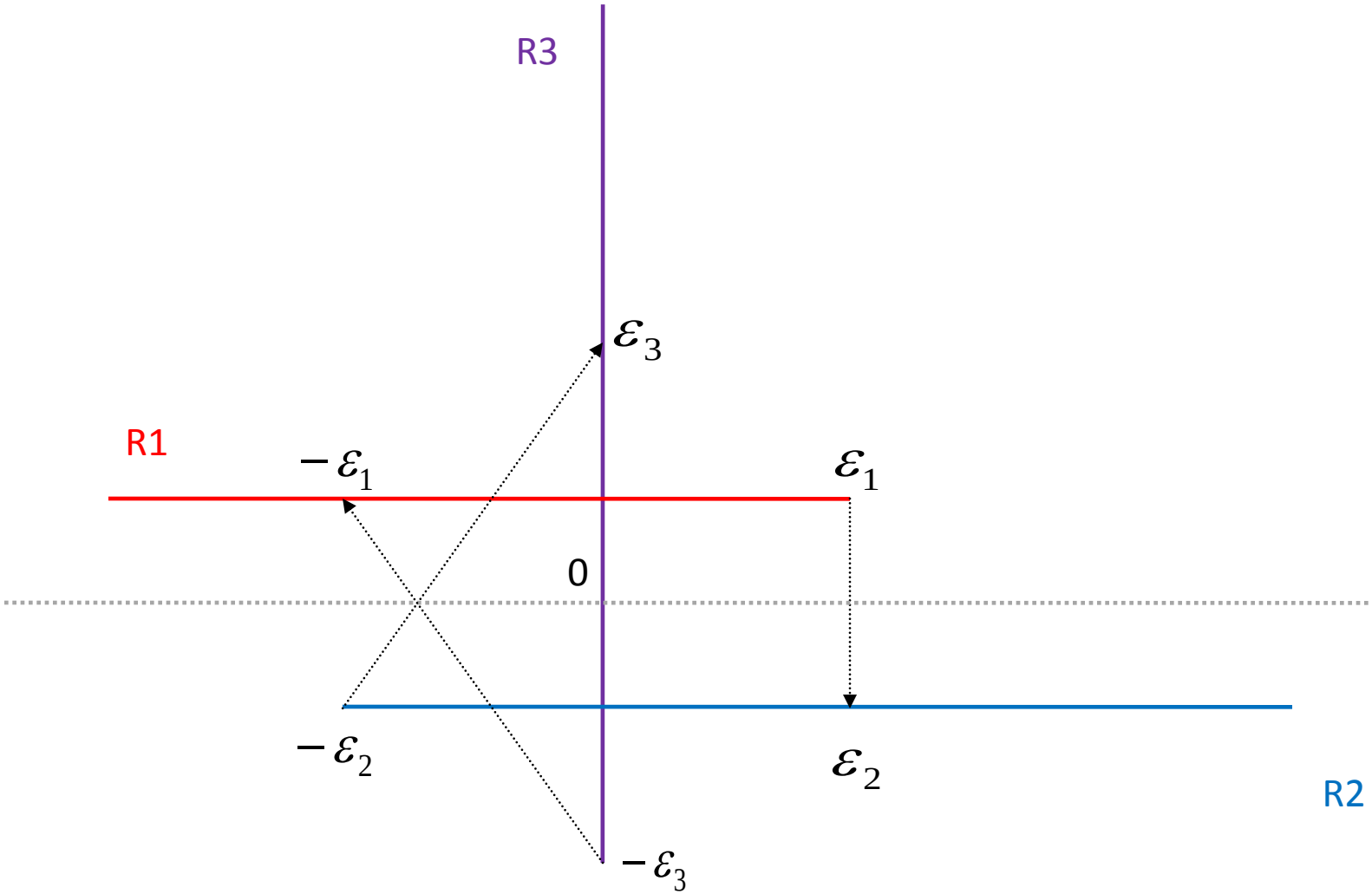
Moreover, also due to controllability, such a limit function uniquely extends to the line $\{x = 0\}$ and it satisfies the problem

$$\begin{cases} \lambda V + H_2(x, \nabla V) = 0 \text{ in } \left(\mathbf{R}^2 \setminus \Gamma\right)_2 \\ \lambda V + H_\Gamma^{reg}(x, \nabla V) = 0 \text{ on } \Gamma \\ \lambda V + H_1(x, \nabla V) = 0 \text{ in } \left(\mathbf{R}^2 \setminus \Gamma\right)_1 \\ \max\{\lambda V + H_1, \lambda V + H_2\} \geq 0 \quad \text{on } \Gamma \\ \min\{\lambda V + H_1, \lambda V + H_2\} \leq 0 \quad \text{on } \Gamma \end{cases}$$

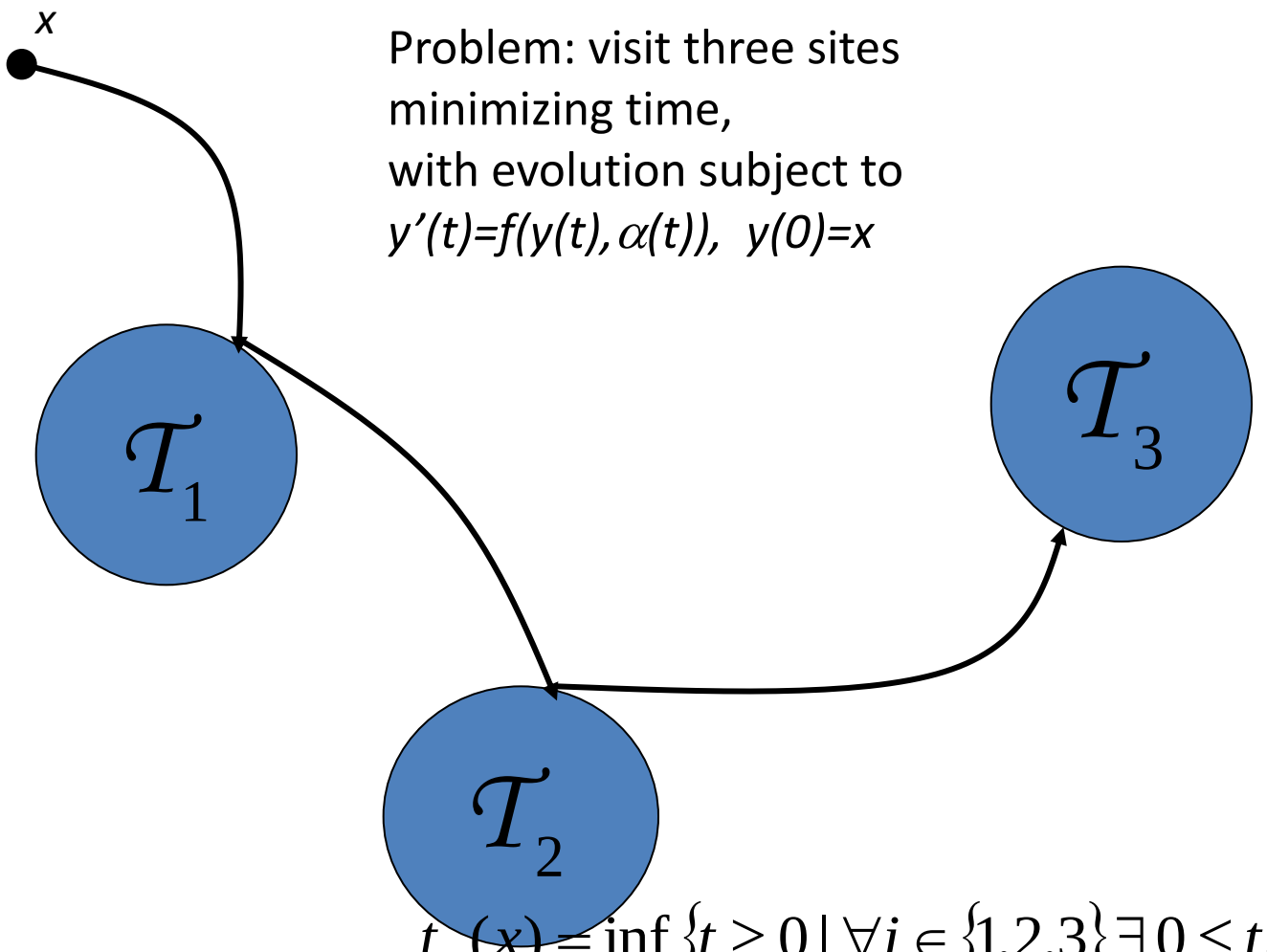
Our guess is that $\tilde{V}$ coincides with the value function U^+ of the regular control problem, that is it is the maximal subsolution.

A one-dimensional three-junction problem




R3
ε_3
R1
$-\varepsilon_1$
ε_1
0
$-\varepsilon_2$
ε_2
R2
$-\varepsilon_3$

Optimal visiting problem

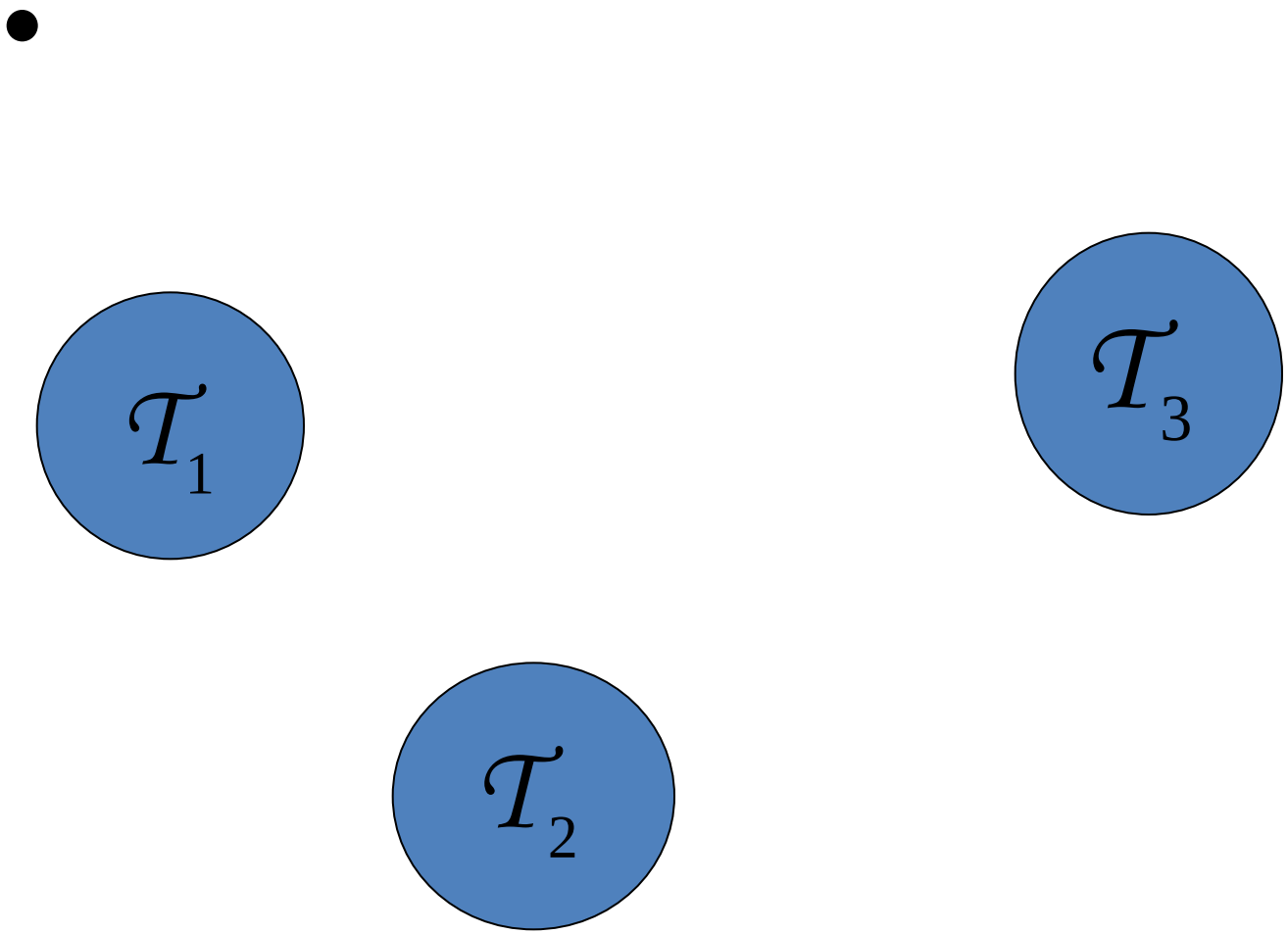


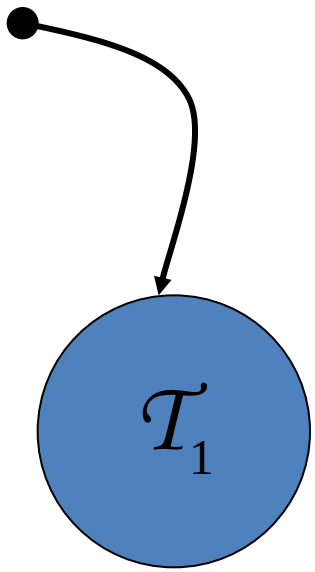
Problem: visit three sites
minimizing time,
with evolution subject to
y'(t)=f(y(t),α(t)), y(0)=x

$$t_\alpha(x) = \inf\left\{t \geq 0 \mid \forall i \in \{1,2,3\} \exists\, 0 \leq t_i \leq t,\, y(t_i) \in \tau_i\right\}$$

$$T(x) = \inf_\alpha t_\alpha(x) \text{ optimal visiting function}$$

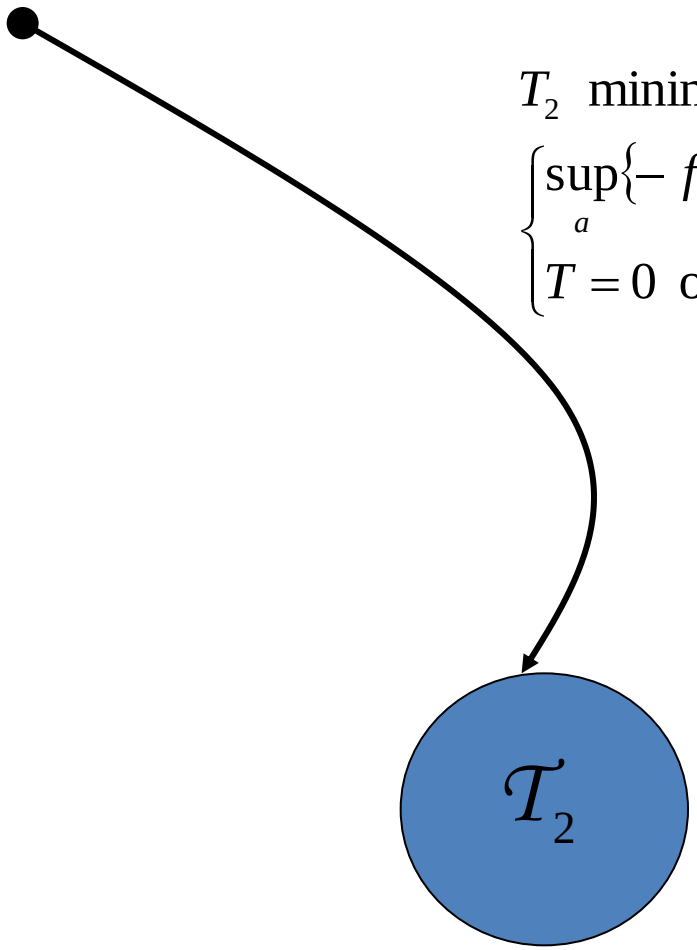
- The problem is obviously reminiscent of the famous Traveling Salesman Problem: minimizing the length of the path for passing through *m* cities.
- It is then characterized by a high computational complexity: many sub-problems must be addressed before solving the initial problem.


$\mathcal{T}_1$
$\mathcal{T}_2$
$\mathcal{T}_3$



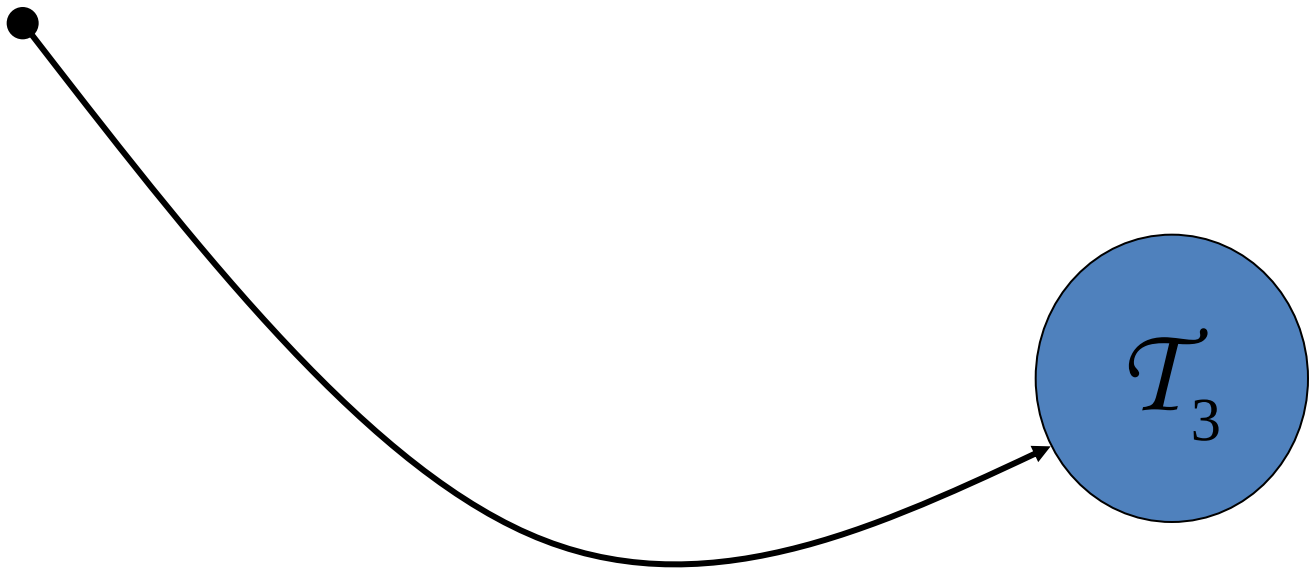
T_1 minimum time function for reaching $\mathcal{T}_1$, which solves

$$\begin{cases} \sup_a \{-f(x,a)\cdot\nabla T(x)\} = 1 \\ T = 0 \text{ on } \partial\mathcal{T}_1 \end{cases}$$



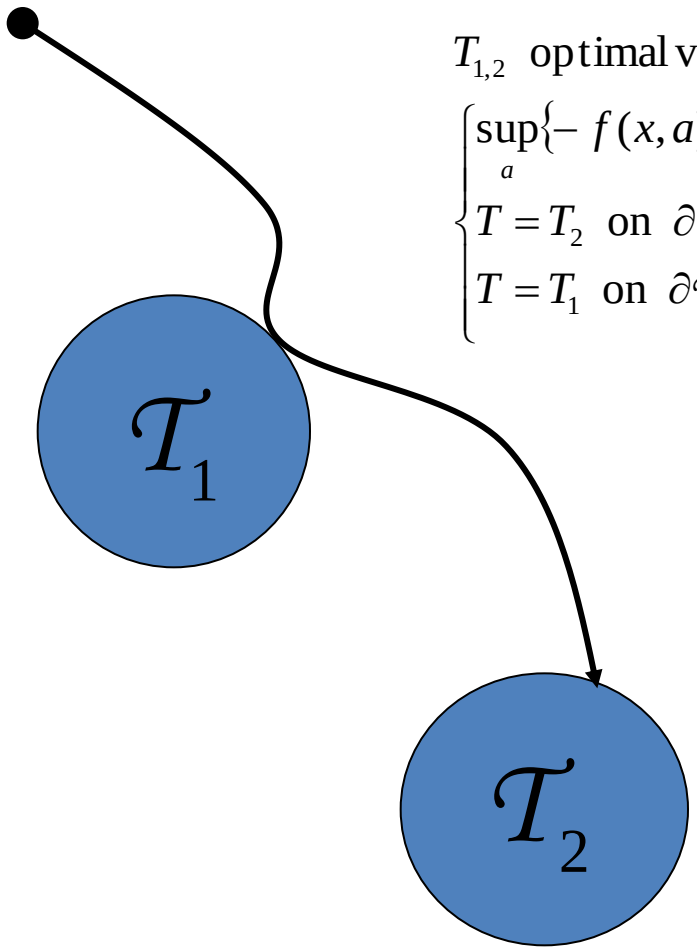
T_2 minimum time function for reaching $\mathcal{T}_2$, which solves

$$\begin{cases} \sup_a \{-f(x,a)\cdot\nabla T(x)\} = 1 \\ T = 0 \text{ on } \partial\mathcal{T}_2 \end{cases}$$



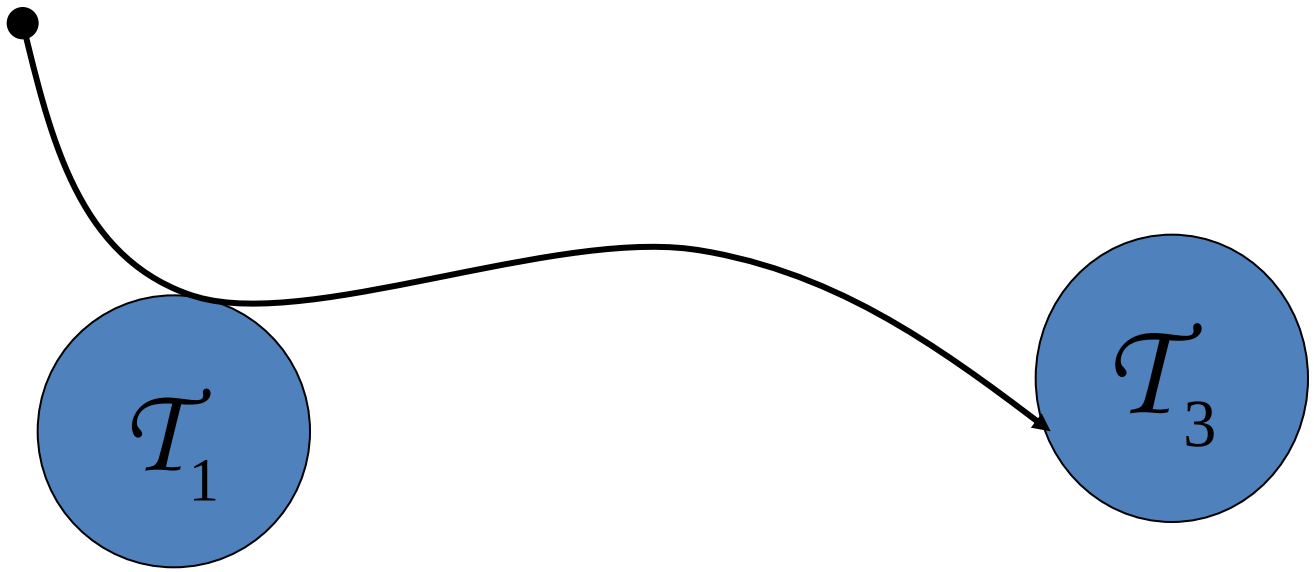
T_3 minimum time function for reaching $\mathcal{T}_3$, which solves

$$\begin{cases} \sup_a \{-f(x,a)\cdot\nabla T(x)\} = 1 \\ T = 0 \text{ on } \partial\mathcal{T}_3 \end{cases}$$



$T_{1,2}$ optimal visting function for reaching $\mathcal{T}_1$ e $\mathcal{T}_2$, which solves

$$\begin{cases} \sup_a \{-f(x,a)\cdot\nabla T(x)\} = 1 \\ T = T_2 \text{ on } \partial\mathcal{T}_1 \\ T = T_1 \text{ on } \partial\mathcal{T}_2 \end{cases}$$

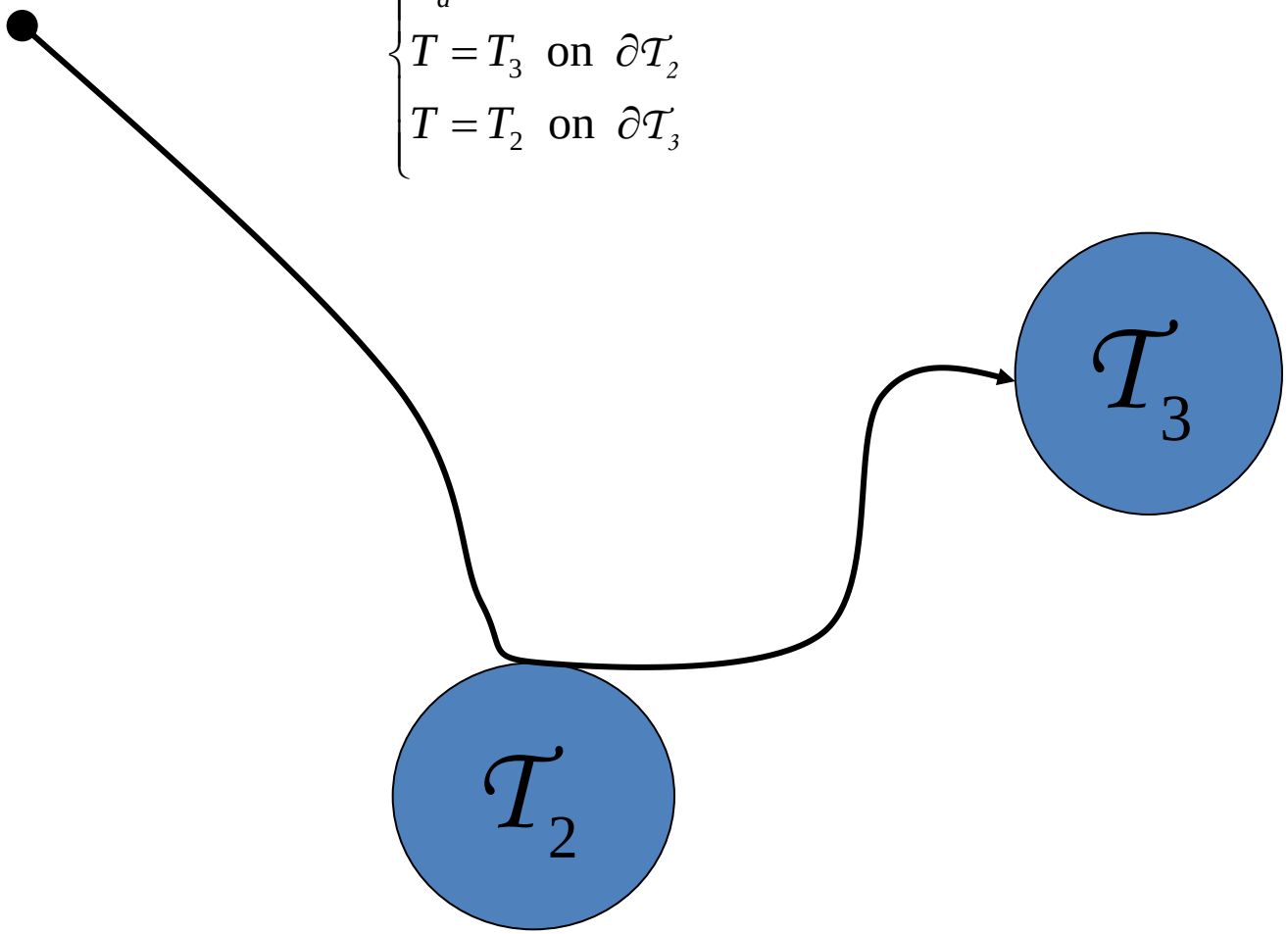


$T_{1,3}$ optimal visiting function for reaching $\mathcal{T}_1$ e $\mathcal{T}_3$, which solves

$$\begin{cases} \sup_a \{-f(x,a)\cdot\nabla T(x)\} = 1 \\ T = T_3 \text{ on } \partial\mathcal{T}_1 \\ T = T_1 \text{ on } \partial\mathcal{T}_3 \end{cases}$$

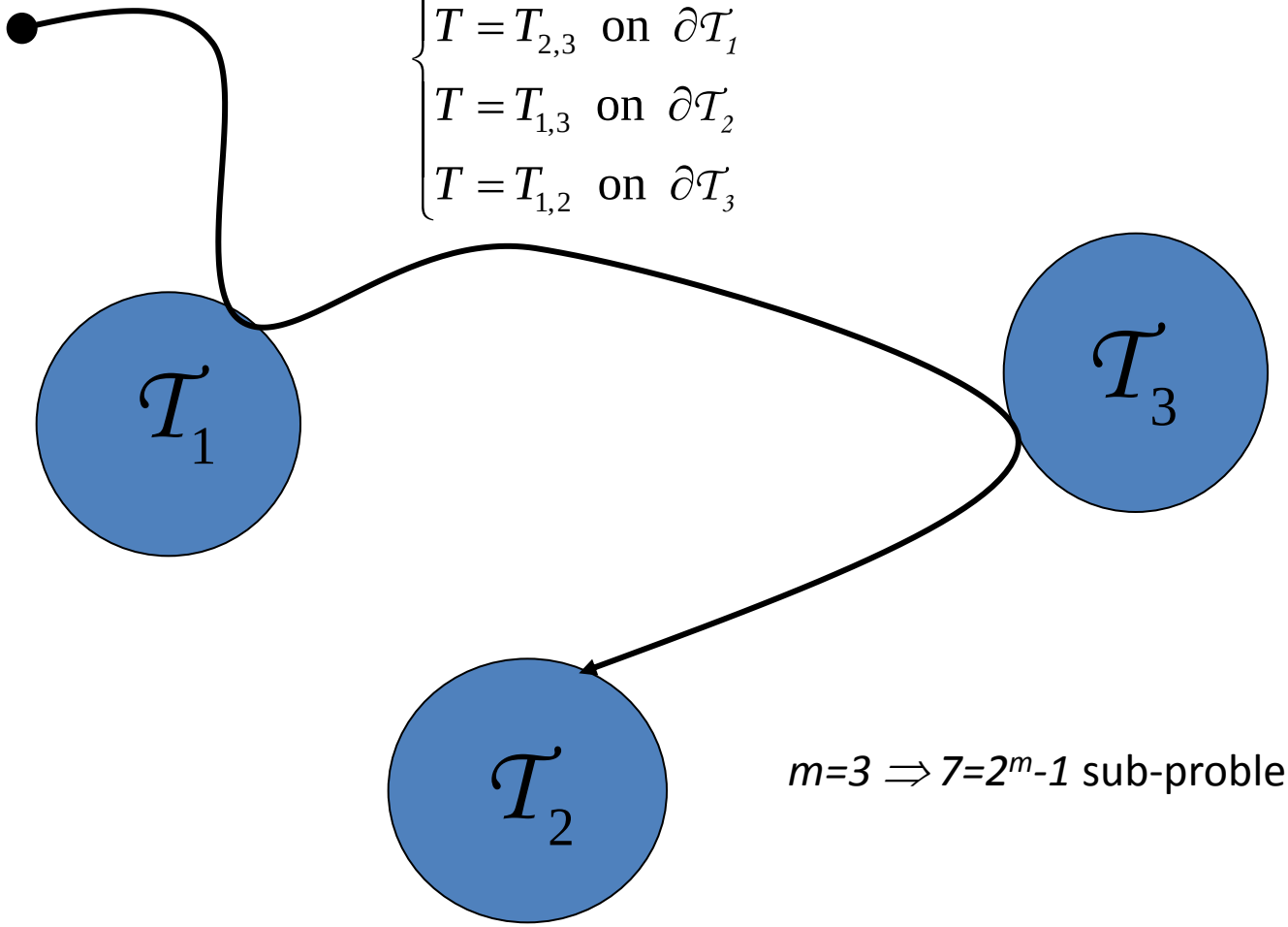
$T_{2,3}$ optimal visiting function for reaching $\mathcal{T}_2$ e $\mathcal{T}_3$, which solves

$$\begin{cases} \sup_a \{ - f(x,a) \cdot \nabla T(x) \} = 1 \\ T = T_3 \text{ on } \partial \mathcal{T}_2 \\ T = T_2 \text{ on } \partial \mathcal{T}_3 \end{cases}$$



$T_{1,2,3}$ optimal visting function for reaching $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$, which solves

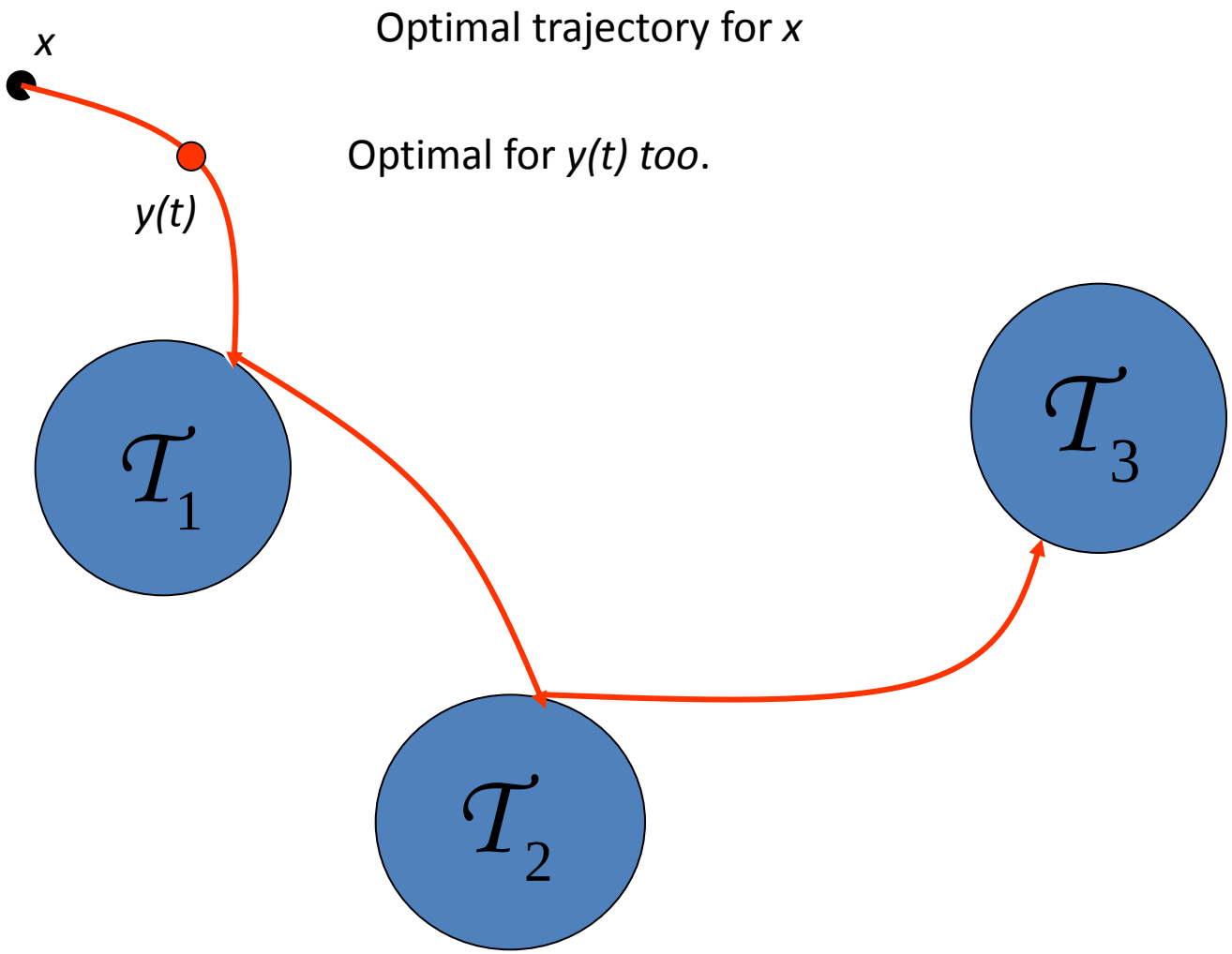
$$\begin{cases} \sup_a \{-f(x,a)\cdot\nabla T(x)\} = 1 \\ T = T_{2,3} \text{ on } \partial\mathcal{T}_1 \\ T = T_{1,3} \text{ on } \partial\mathcal{T}_2 \\ T = T_{1,2} \text{ on } \partial\mathcal{T}_3 \end{cases}$$

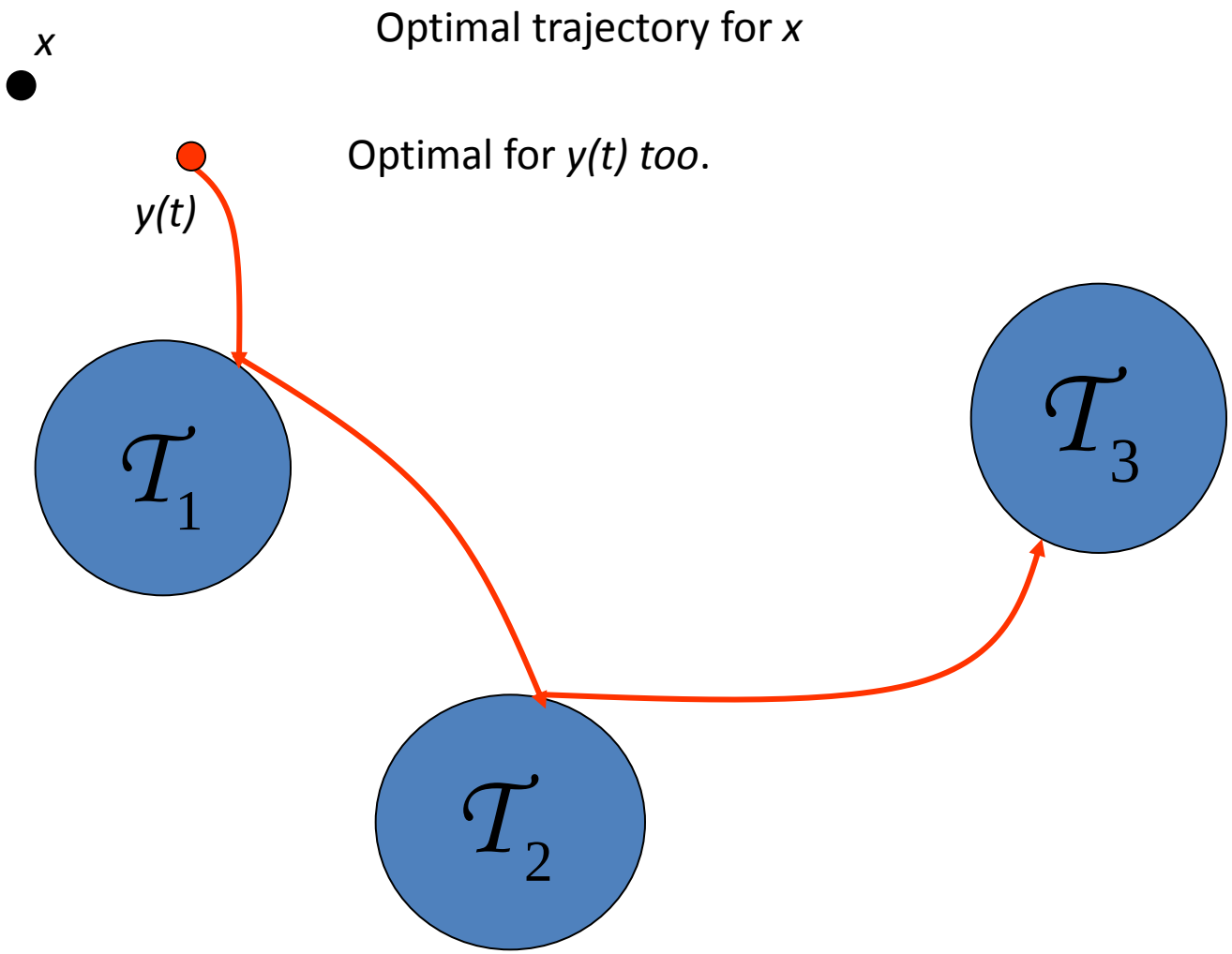


$m=3 \Rightarrow 7=2^m-1$ sub-problems

Goal

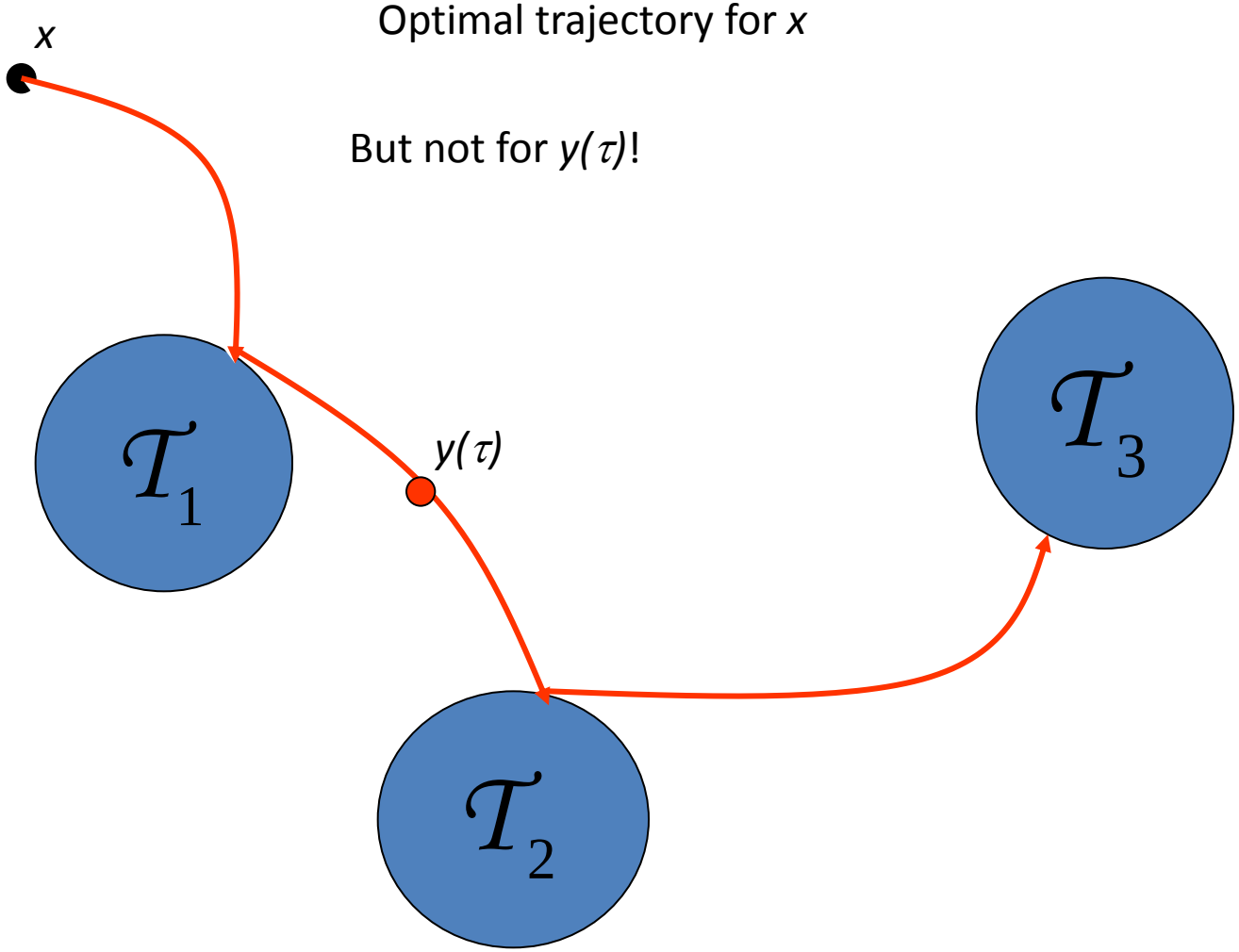
- Use Dynamic Programming for writing a "single" equation uniquely satisfied by the optimal visiting function.
- An immediate problem:
- The Dynamic Programming Principle does not hold.
- "Pieces of optimal trajectories are not optimal"!


Optimal trajectory for *x*
Optimal for *y(t) too*.
x
y(t)
$\mathcal{T}_1$
$\mathcal{T}_2$
$\mathcal{T}_3$


Optimal trajectory for *x*
x
Optimal for *y(t) too*.
y(t)
$\mathcal{T}_1$
$\mathcal{T}_2$
$\mathcal{T}_3$

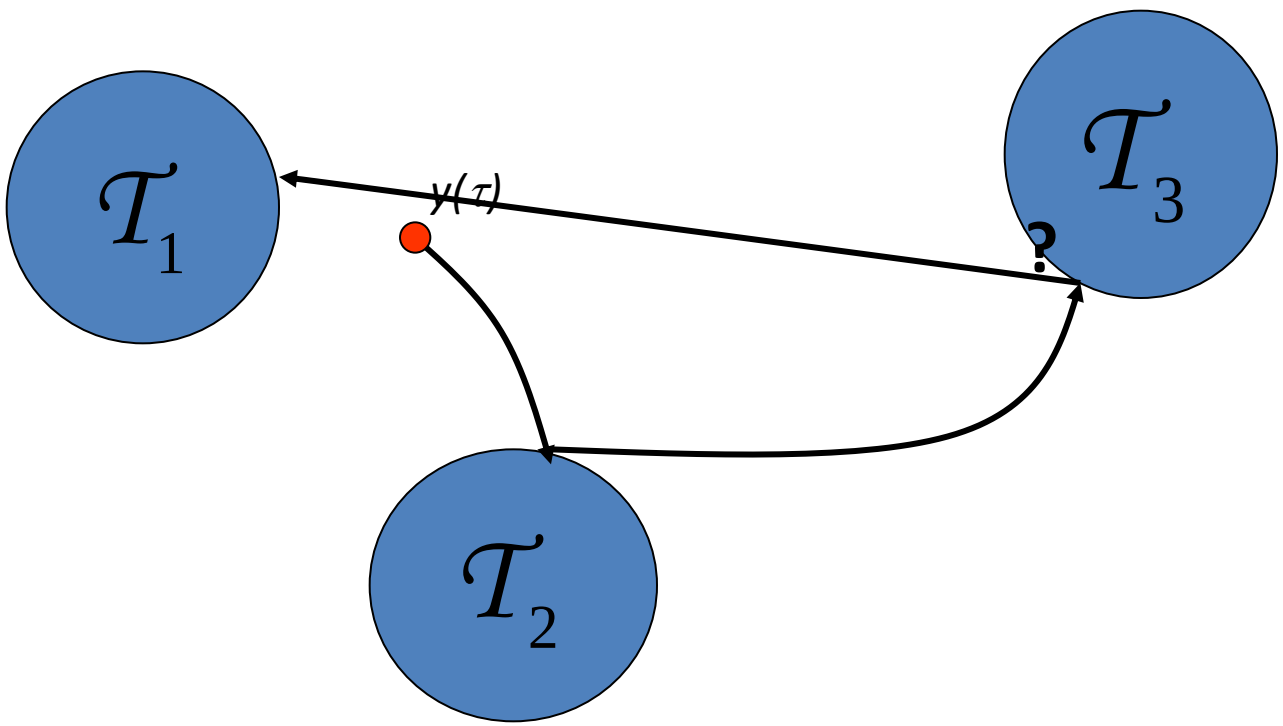
Optimal trajectory for x

But not for $y(\tau)$!



Optimal trajectory for *x*

But not for *y(τ)*!

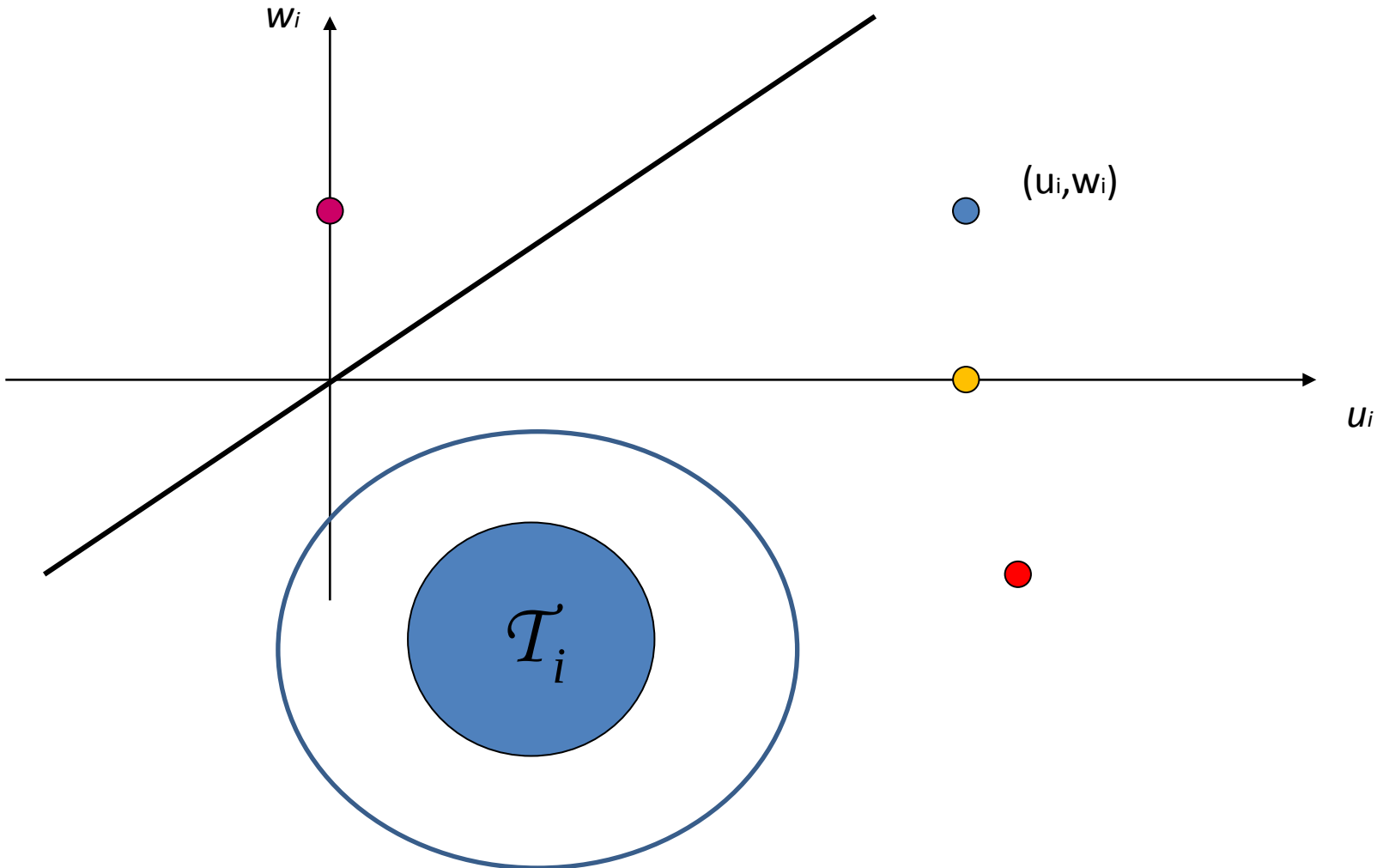


Need of memory

- We need a sort of memory!
- We have to keep in mind whether the *i*-th target is already visited or not.
- For every *i,* we need a positive scalar w_i, evolving in time, which is zero if and only if we have already reached the *i*-th target.

Hysteresis operator

$$u_i(t) = dist\big(y(t), \mathcal{T}_i\big), \qquad w_i(t) = \min_{\tau \in [0,t]} \Big(dist\big(y(\tau), \mathcal{T}_i\big)\Big) = \min_{\tau \in [0,t]} u_i(\tau)$$



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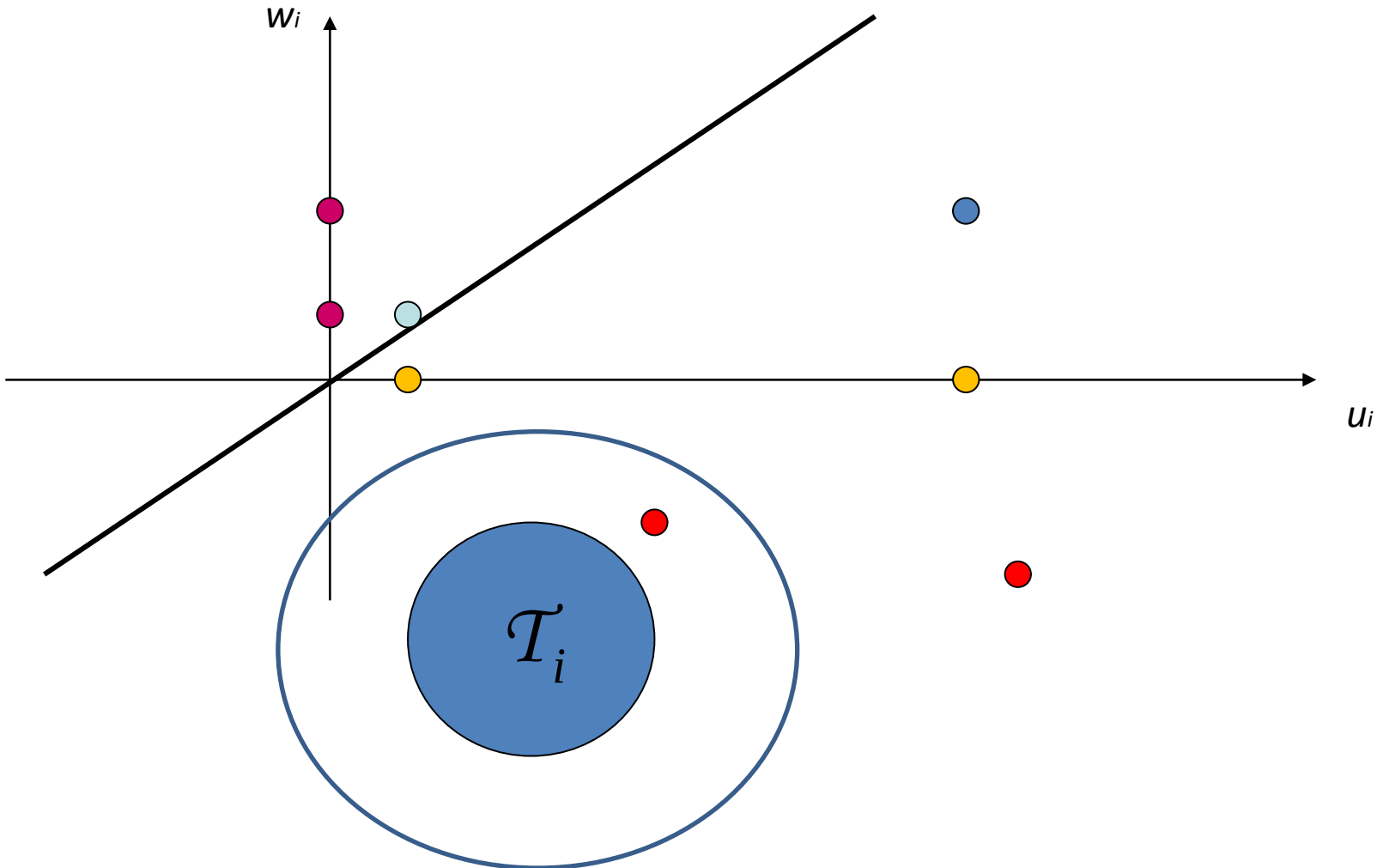

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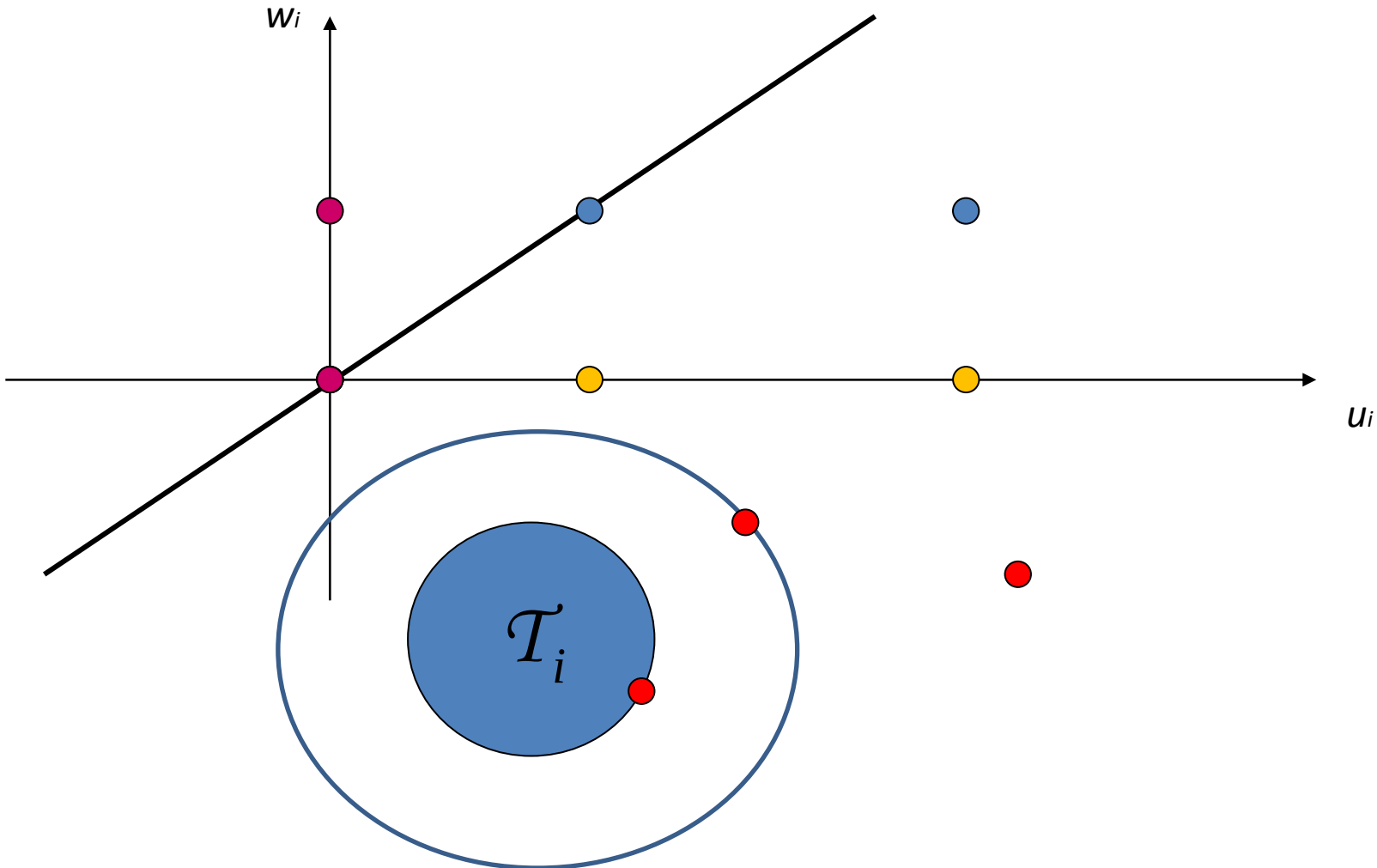

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w_i can be written as the unique solution of

a discontinuos ODE depending on u_i as parameter;

or as solution of variational inequality, also depending on u_i.

Need of memory

- We need a sort of memory!
- We have to keep in mind whether the i-th target is already visited or not.
- For every i, we need a positive scalr w_i, evolving in time, which is zero if and only if we have already reached the i-th target.
- The new state variable is $(x, w_1, w_2, \ldots, w_m)$ and the new target is $\{w_1 = w_2 = \ldots = w_m = 0\}$.
- We recover the Dynamic Programming Principle and we can write a (discontinuous) Hamilton-Jacobi equation uniquely satisfied by the optimal visiting function.
- We pay a price: we add more variables (the w_i) and the equation is discontinuous.
- A result in this sense was already obtained in the case when the sequence of visited targets is free and at the end it is not mandatory to return to the initial state (as indeed is for the classical Salesman Travelling Problem)