On linear output regulation:
data-driven, hybrid, optimal, and more!

Sergio Galeani

joint work with
(data-driven) D. Carnevale, M. Sassano, A. Serrani

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Outline

1. The output regulation problem: Internal vs External Model
2. Data-driven External Model based regulators: the non hybrid case
3. An example for the non hybrid case
4. Conclusions
Section 1

The output regulation problem: Internal vs External Model
Problem (Output regulation)

Find, if possible, an error feedback compensator \( C \) such that

- the closed loop system is exponentially stable
- the output \( e \) is asymptotically regulated to zero

Several flavours:

- full information vs. error feedback
- hybrid dynamics
- known vs unknown plant and/or exosystem
- dealing with fat plants: control allocation, optimal steady-state maps
The output regulation problem

Problem (Output regulation)

Find, if possible, an error feedback compensator $C$ such that

- the closed loop system is exponentially stable
- the output $e$ is asymptotically regulated to zero

Key observations:

1. when using error feedback, steady-state is in open-loop
2. $C$ makes $E$ unobservable from $e$
3. by linearity, stability and regulation can be studied separately
Problem (Output regulation via External Models for Unknown Plants)

Find, if possible, a data-driven compensator $E_M$ and Reset Logic such that:

1. The closed loop system is exponentially stable.
2. The output is asymptotically regulated to zero.

Without any knowledge of the (pre-stabilized) plant $P$, subject to arbitrary parameter uncertainties.

NOTE: a robust output regulation problem!

Dynamics of (Feed-forward) compensator $E_M$ based on the Internal Model Principle!
Problem (Output regulation via External Models for Unknown Plants)

Find, if possible, a data-driven compensator $\mathcal{E}_M$ and Reset Logic such that

- the closed loop system is exponentially stable
- the output is asymptotically regulated to zero
- without any knowledge of the (pre-stabilized) plant $\mathcal{P}$ subject to arbitrary parameter uncertainties

**NOTE:** a robust output regulation problem!

Dynamics of (Feed-forward) compensator $\mathcal{E}_M$ based on the Internal Model Principle!
A look forward: three steps towards data-driven OR

- $u = 0$: learn the effect $\bar{e}$ of $w$ on $e$
- Random reset of $\mathcal{E}_M$: learn the effect of $u$ on $\bar{e} := e - \bar{e}$
- Smart reset $\mathcal{E}_M$: achieve OR!

\[ \mathcal{E} \]

$w \rightarrow u \rightarrow \mathcal{P} \rightarrow \mathcal{P}_0 \rightarrow \mathcal{K} \rightarrow e$

\[ \mathcal{E}_M \]

\[ \text{Reset logic} \]
Classic setting and Francis equations

- **LTI exosystem** \( \mathcal{E} \): \( \dot{w} = Sw \)
- **LTI plant** \( \mathcal{P}_0 \):
  \[
  \begin{aligned}
  \dot{x} &= Ax + Bu + Pw, \\
  e &= Cx + Du + Qw
  \end{aligned}
  \]

**Assumptions**

1. \( S \) is semi-simple with \( \text{spec}(S) = \{0, \pm j\omega_1, \ldots, \pm j\omega_r\} \)
2. \((A, B)\) stabilizable, \((A, C)\) detectable
3. \( \text{rank} \begin{bmatrix} A - j\omega_h & B \\ C & D \end{bmatrix} = n + p, \ h = 0, 1, \ldots, r, \) satisfied by all \((A, B, C, D)\)

For properly designed \( \mathcal{K}, \mathcal{I}_M \):

- **Well defined steady-state solution:** \[
  \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} w
  \]
- **Francis equations:** (invariance and internal model)
  \[
  \begin{aligned}
  \Pi S &= A\Pi + B\Gamma + P \\
  \Sigma S &= F\Sigma \\
  0 &= C\Pi + D\Gamma + Q \\
  \Gamma &= H\Sigma
  \end{aligned}
  \]

where \((F, G, H, 0)\) are the matrices of the overall error feedback controller
Section 2

Data-driven External Model based regulators: the non hybrid case
LTI exosystem $\mathcal{E}$: $\dot{w} = Sw$

LTI plant $\mathcal{P}$:
$$\begin{cases}
\dot{x} = Ax + Bu + Pw, \\
e = Cx + Du + Qw
\end{cases}$$

LTI external model $\mathcal{E}_M$:
$$\begin{cases}
\dot{x}_M = A_M x_M, & A_M = I_p \otimes A_m, \\
y_M = C_M x_M, & C_M = F(I_p \otimes C_m), \\
\end{cases}$$
with $(A_m, C_m)$ observable and $\mu_{A_m}(s) = \mu_S(s)$

**Assumptions**

1. $S$ is semi-simple with $\text{spec}(S) = \{0, \pm j\omega_1, \ldots, \pm j\omega_r\}$
2. $\text{spec}(A) \subset \mathbb{C}_g$
3. $\text{rank}\begin{bmatrix} A - j\omega_h & BF \\ C & DF \end{bmatrix} = n + p$, $h = 0, 1, \ldots, r$, satisfied by all $(A, B, C, D)$
External model - preliminary design

- **LTI exosystem** $\mathcal{E}$: $\dot{w} = Sw$

- **LTI plant** $\mathcal{P}$:
  \[
  \begin{align*}
  \dot{x} &= Ax + Bu + Pw, \\
  e &= Cx + Du + Qw
  \end{align*}
  \]

- **LTI external model** $\mathcal{E}_M$:
  \[
  \begin{align*}
  \dot{x}_M &= A_M x_M, & A_M &= I_p \otimes A_m, \\
  y_M &= C_M x_M, & C_M &= F(I_p \otimes C_m),
  \end{align*}
  \]
  with $(A_m, C_m)$ observable and $\mu_{A_m}(s) = \mu_S(s)$

---

**Ignoring the Reset logic:**

- **Well defined steady-state solution:**
  \[
  \begin{bmatrix}
  x_{ss} \\
  u_{ss}
  \end{bmatrix} = \begin{bmatrix}
  \Pi_M \\
  C_M
  \end{bmatrix} x_M + \begin{bmatrix}
  \Pi_w \\
  0
  \end{bmatrix} w
  \]
  \[
  \Pi_M A_M = A \Pi_M + B C_M, \quad \Pi_w S = A \Pi_w + P,
  \]

- **Steady-state error:** $e_{ss} = M x_M + N w$ where $M = C \Pi_M + D C_M$, $N = C \Pi_w + Q$

- $e \equiv 0$ iff $x_M(0) = \Psi w(0)$ with $\Psi$ such that
  \[
  \Psi S = A_M \Psi \\
  0 = M \Psi + N
  \]
A coordinate transformation

- For the interconnection of $\mathcal{E}_M$, $\mathcal{E}$ and $\mathcal{P}$, change coordinates to expose:
  - the offset $z$ between $x$ and its steady-state value $x_{ss}$: $z = x - \Pi_M x_M - \Pi w w$
  - the offset $\eta$ between $x_M$ and its ideal value $\Psi w$: $\eta = x_M - \Psi w$

Hence, for $t$ large enough, it holds that $e(t) \approx M \eta(t)$.
A coordinate transformation

- For the interconnection of \( E_M, E \) and \( P \), **change coordinates** to expose:
  - the offset \( z \) between \( x \) and its steady-state value \( x_{ss} \): \( z = x - \Pi_M x_M - \Pi_w w \)
  - the offset \( \eta \) between \( x_M \) and its ideal value \( \Psi w \): \( \eta = x_M - \Psi w \)

- It can be easily computed that
  \[
  \begin{bmatrix}
  \dot{w} \\
  \dot{z} \\
  \dot{\eta} \\
  \dot{e}
  \end{bmatrix}
  =
  \begin{bmatrix}
  Sw \\
  Az \\
  A_M \eta \\
  Cz + M\eta
  \end{bmatrix}, \quad \text{since:}
  \]

  \[
  \dot{z} = \dot{x} - \Pi_M \dot{x}_M - \Pi_w \dot{w}
  = (Ax + BC_M x_M + Pw) - \Pi_M A_M x_M - \Pi_w Sw
  = Az + (A\Pi_M + BC_M - \Pi_M A_M)x_M + (A\Pi_w + P - \Pi_w S)w = Az,
  \]

  \[
  \dot{\eta} = \dot{x}_M - \Psi \dot{w}
  = A_M x_M - \Psi Sw
  = A_M \eta + (A_M \Psi - \Psi S)w = A_M \eta,
  \]

  \[
  e = \cdots = Cz + M\eta.
  \]
A coordinate transformation

- For the interconnection of $E_M$, $E$ and $P$, change coordinates to expose:
  - the offset $z$ between $x$ and its steady-state value $x_{ss}$: $z = x - \Pi_M x_M - \Pi_w w$
  - the offset $\eta$ between $x_M$ and its ideal value $\Psi w$: $\eta = x_M - \Psi w$

- It can be easily computed that
  \[
  \begin{bmatrix}
  \dot{w} \\
  \dot{z} \\
  \dot{\eta} \\
  e
  \end{bmatrix} =
  \begin{bmatrix}
  Sw \\
  Az \\
  A_M \eta \\
  Cz + M\eta
  \end{bmatrix},
  \]
  since:
  \[
  \dot{z} = \dot{x} - \Pi_M \dot{x}_M - \Pi_w \dot{w}
  = (Ax + BC_M x_M + Pw) - \Pi_M A_M x_M - \Pi_w Sw
  = Az + (A\Pi_M + BC_M - \Pi_M A_M)x_M + (A\Pi_w + P - \Pi_w S)w = Az,
  \]
  \[
  \dot{\eta} = \dot{x}_M - \Psi \dot{w}
  = A_M x_M - \Psi Sw
  = A_M \eta + (A_M \Psi - \Psi S)w = A_M \eta,
  \]
  \[
  e = \cdots = Cz + M\eta.
  \]

- Hence, for $t$ large enough, it holds that $e(t) \approx M\eta(t)$

Regulation with a single reset is achieved by “inverting” this relation!
Inverting the error relation, when $M$ is known

- Let $T, \tau_0 \in \mathbb{R}_{>0}$ with $\tau_0 \ll T$, and $A_{MD} = e^{AMT}$, $A_{MI} = e^{-AM\tau_0}$, $A_D = e^{AT}$, $A_I = e^{-A\tau_0}$. For $h \in \mathbb{N}$, define the extended error $\hat{e}(t)$

$$
\hat{e}(t) = \begin{bmatrix}
e(t) \\
e(t - \tau_0) \\
\vdots \\
e(t - h\tau_0)
\end{bmatrix} = C_D z(t - T) + M_D \eta(t - T),
$$

where $\lim_{T \to +\infty} A_D = \lim_{T \to +\infty} e^{AT} = 0$ and $\lim_{T \to +\infty} C_D = 0$
Inverting the error relation, when $M$ is known

- Let $T, \tau_0 \in \mathbb{R}_{>0}$ with $\tau_0 \ll T$, and $A_{MD} = e^{AMT}, A_{MI} = e^{-AM\tau_0}, A_D = e^{AT}, A_I = e^{-A\tau_0}$.

For $h \in \mathbb{N}$, define the extended error $\hat{e}(t)$

$$\hat{e}(t) = \begin{bmatrix} e(t) \\ e(t - \tau_0) \\ \vdots \\ e(t - h\tau_0) \end{bmatrix} = C_D z(t - T) + M_D \eta(t - T),$$

$$C_D := \begin{bmatrix} C \\ CA_I \\ \vdots \\ CA_I^{h-1} \end{bmatrix} A_D, \quad M_D := \begin{bmatrix} M \\ MA_{MI} \\ \vdots \\ MA_{MI}^{h-1} \end{bmatrix} A_{MD},$$

where $\lim_{T \to +\infty} A_D = \lim_{T \to +\infty} e^{AT} = 0$ and $\lim_{T \to +\infty} C_D = 0$

- Large $T$ and $t > T$ imply $\hat{e}(t) \approx M_D \eta(t - T)$ with $M_D$ left invertible, so that

$$\eta(t - T) \approx M_D^\dagger \hat{e}(t), \quad \eta(t) \approx A_{MD} \eta(t - T)$$
Inverting the error relation, when $M$ is known

- Let $T, \tau_0 \in \mathbb{R}_{>0}$ with $\tau_0 \ll T$, and $A_{MD} = e^{A_M T}$, $A_{MI} = e^{-A_M \tau_0}$, $A_D = e^{A T}$, $A_I = e^{-A \tau_0}$.

For $h \in \mathbb{N}$, define the extended error $\hat{e}(t)$

$$\hat{e}(t) = \begin{bmatrix} e(t) \\ e(t - \tau_0) \\ \vdots \\ e(t - h\tau_0) \end{bmatrix} = C_D z(t - T) + M_D \eta(t - T),$$

$$C_D := \begin{bmatrix} C \\ CA_I \\ \vdots \\ CA_I^{h-1} \end{bmatrix} A_D, \quad M_D := \begin{bmatrix} M \\ MA_{MI} \\ \vdots \\ MA_{MI}^{h-1} \end{bmatrix} A_{MD},$$

where $\lim_{T \to +\infty} A_D = \lim_{T \to +\infty} e^{A T} = 0$ and $\lim_{T \to +\infty} C_D = 0$

- Large $T$ and $t > T$ imply $\hat{e}(t) \approx M_D \eta(t - T)$ with $M_D$ left invertible, so that

$$\eta(t - T) \approx M_D^\# \hat{e}(t), \quad \eta(t) \approx A_{MD} \eta(t - T)$$

- Reset $x_M(t^+) := x_M(t^-) - A_{MD} M_D^\# \hat{e}(t)$ to achieve $x_M(t^+) \approx \Psi w(t)$ since

$$x_M(t^+) \approx x_M(t^-) - \eta(t^-) = x_M(t^-) - (x_M(t^-) + \Psi w(t)) = \Psi w(t)$$

A single reset yields practical regulation, for known $M$.
Asymptotic regulation with $M$ unknown

- Let $\eta[k] = \eta(kT)$, $\eta_{\langle k \rangle} = \eta(kT^{-})$, etc
- Define the reset rule for $x_M$ as
  \[ x_{M,[k]} = x_{M,\langle k \rangle} + L \hat{\eta}_{\langle k \rangle}, \text{ where } \hat{\eta}_{\langle k \rangle} = C_D z_{[k-1]} + M_D \eta_{[k-1]} \]
- It can be shown that
  \[
  \begin{bmatrix}
  z_{[k+1]} \\
  \eta_{[k+1]}
  \end{bmatrix} =
  \begin{bmatrix}
  A_D - \Pi_M L C_D & -\Pi_M L M_D \\
  L C_D & A_{MD} + L M_D
  \end{bmatrix}
  \begin{bmatrix}
  z_{[k]} \\
  \eta_{[k]}
  \end{bmatrix}
  \]
  (1)

  where $\lim_{T \to +\infty} A_D = 0$ and $\lim_{T \to +\infty} C_D = 0$
Asymptotic regulation with $M$ unknown

Let $\eta[k] = \eta(kT)$, $\eta_\langle k \rangle = \eta(kT^-)$, etc.

Define the reset rule for $x_M$ as

$$x_M[k] = x_M_\langle k \rangle + L\hat{\eta}_\langle k \rangle,$$

where $\hat{\eta}_\langle k \rangle = CDz[k-1] + MD\eta[k-1]$.

It can be shown that

$$\begin{bmatrix} z[k+1] \\ \eta[k+1] \end{bmatrix} = \begin{bmatrix} A_D - \Pi_MLD & -\Pi_MLM_D \\ LC_D & A_{MD} + LM_D \end{bmatrix} \begin{bmatrix} z[k] \\ \eta[k] \end{bmatrix}$$

where $\lim_{T \to +\infty} A_D = 0$ and $\lim_{T \to +\infty} C_D = 0$.

Proposition

As $T \to +\infty$, the spectrum of the matrix in (1) approaches the set $\{0\} \cup \text{spec}(A_{MD} + LM_D)$.

Moreover, let $\hat{M}_D$ be such that $\lim_{T \to +\infty} (\hat{M}_D - M_D) = 0$. If $L$ is such that $(A_{MD} + L\hat{M}_D)$ is Schur, then there exists $T_1^* > 0$ such that for any $T > T_1^*$ the equilibrium $(z_{eq}, \eta_{eq}) = (0,0)$ of (1) is globally exponentially stable.

A periodic reset yields asymptotic regulation, for unknown $M$. 
Estimating $M$ (when $w$ is not there)

- Consider a **scalar input** $u$, the system
  \[
  \begin{align*}
  \dot{x} &= Ax + Bu, \\
  \bar{e} &= Cx + Du,
  \end{align*}
  \]
  and
  \[
  \begin{align*}
  \dot{x}_m &= A_m x_m, \\
  y_m &= C_m x_m = u,
  \end{align*}
  \]

- Choose $\tau_1 > 0$ such that
  \[
  \tau_1 \neq \frac{2\pi}{\omega_i - \omega_j}, \quad \forall i, j \in \{0, 1, \ldots, r\},
  \]
  where $\omega_0 = 0$

- Choose $z \in \mathbb{N}$, $z \geq q$ and define
  \[
  M_a(t) := \begin{bmatrix}
  \bar{e}(t) & \bar{e}(t - \tau_1) & \bar{e}(t - 2\tau_1) & \cdots & \bar{e}(t - z\tau_1)
  \end{bmatrix},
  \]
  \[
  \dot{M}_b(t) := A_m M_b(t),
  \]
  \[
  M_b(0) := \begin{bmatrix}
  x_{m0} & \tilde{A}_m^{-1} x_{m0} & \tilde{A}_m^{-2} x_{m0} & \cdots & \tilde{A}_m^{-z} x_{m0}
  \end{bmatrix},
  \]
  \[
  \hat{M}(t) := M_a(t) M_b^\#(t),
  \]

  where $\tilde{A}_m := e^{A_m \tau_1}$ and $x_{m0}$ is such that $(A_m, x_{m0})$ is reachable.

- Recalling that $z := x - x_{ss}$, it is easy to see that
  \[
  M_b(t) = \begin{bmatrix}
  x_m(t) & x_m(t - \tau_1) & x_m(t - 2\tau_1) & \cdots & x_m(t - z\tau_1)
  \end{bmatrix}
  \]
  \[
  M_a(t) = MM_b(t) + C \begin{bmatrix}
  z(t) & z(t - \tau_1) & z(t - 2\tau_1) & \cdots & z(t - z\tau_1)
  \end{bmatrix}
  \]

Proposition

For any $\varepsilon_M > 0$ there exists $T^* > 0$ such that if $T > T^*$ then
  \[
  \|\hat{M}(T) - M\| < \varepsilon_M.
  \]
Estimating $M$ (when $w$ is not there)

- Consider a **scalar input** $u$, the system
  \[
  \begin{align*}
  \dot{x} &= Ax + Bu, \\
  \bar{e} &= Cx + Du,
  \end{align*}
  \]
  and
  \[
  \begin{align*}
  \dot{x}_m &= A_m x_m, \\
  y_m &= C_m x_m = u,
  \end{align*}
  \]

- Choose $\tau_1 > 0$ such that $\tau_1 \neq \frac{2\pi}{\omega_i - \omega_j}$, $\forall i, j \in \{0, 1, \ldots, r\}$, where $\omega_0 = 0$

- Choose $z \in \mathbb{N}$, $z \geq q$ and define
  \[
  M_a(t) := \begin{bmatrix} \bar{e}(t) & \bar{e}(t-\tau_1) & \bar{e}(t-2\tau_1) & \cdots & \bar{e}(t-z\tau_1) \end{bmatrix},
  \]
  \[
  \dot{M}_b(t) := A_m M_b(t),
  \]
  \[
  M_b(0) := \begin{bmatrix} x_m(0) & \tilde{A}_m^{-1} x_m(0) & \tilde{A}_m^{-2} x_m(0) & \cdots & \tilde{A}_m^{-z} x_m(0) \end{bmatrix},
  \]
  \[
  \dot{\hat{M}}(t) := M_a(t) M_b^\#(t),
  \]
  where $\tilde{A}_m := e^{A_m \tau_1}$ and $x_m(0)$ is such that $(A_m, x_m(0))$ is reachable

- Recalling that $z := x - x_{ss}$, it is easy to see that
  \[
  M_b(t) = \begin{bmatrix} x_m(t) & x_m(t-\tau_1) & x_m(t-2\tau_1) & \cdots & x_m(t-z\tau_1) \end{bmatrix},
  \]
  \[
  M_a(t) = MM_b(t) + C \begin{bmatrix} z(t) & z(t-\tau_1) & z(t-2\tau_1) & \cdots & z(t-z\tau_1) \end{bmatrix}
  \]

Proposition

For any $\varepsilon_M > 0$ there exists $T_3^* > 0$ such that if $T > T_3^*$ then $\hat{M}(T) - M$ has norm less than $\varepsilon_M$.

If $u$ is not scalar, $M$ is computed iteratively (columnwise)
Offsetting the disturbance

Original plant and exosystem

\[
\begin{align*}
\dot{w} &= Sw, \\
\dot{x} &= Ax + Bu + Pw, \\
e &= Cx + Du + Qw,
\end{align*}
\]

Equivalent plant and exosystem

\[
\begin{align*}
\dot{w} &= Sw, \\
S &= I_p \otimes A_m, \\
\dot{x} &= Ax + Bu, \\
P &= 0, \\
e &= Cx + Du + Qw, \\
Q &= I_p \otimes C_m,
\end{align*}
\]

- The steady-state of \(e\) for \(u \equiv 0\) is “matched” by the equivalent output disturbance generator:

\[
\begin{align*}
\dot{x}_{ed}(t) &= A_{ed}x_{ed}(t) + \delta(t)L_{ed}(e(t) - \tilde{e}(t)), \\
\tilde{e}(t) &= C_{ed}x_{ed}(t), \\
C_{ed} &= I_p \otimes C_m, \\
L_m : \text{spec}(A_m - L_mC_m) &\subset \mathbb{C}_g, \text{ [chosen for fast decay of the estimation error]} \\
\delta(t) &= \begin{cases} 
0, & \text{if } \max_{t \in [t-T,t]} |e(t) - \tilde{e}(t)| < \varepsilon_\nu, \\
1, & \text{otherwise},
\end{cases}
\end{align*}
\]

- Generate the “(almost) disturbance free” output as \(\bar{e} = e - \tilde{e}\)
Offsetting the disturbance

Original plant and exosystem

\begin{align*}
\dot{w} &= Sw, \\
\dot{x} &= Ax + Bu + Pw, \\
e &= Cx + Du + Qw,
\end{align*}

Equivalent plant and exosystem

\begin{align*}
\dot{w} &= Sw, \\
\dot{x} &= Ax + Bu, \\
e &= Cx + Du + Qw,
\end{align*}

\begin{align*}
S &= Ip \otimes A_m, \\
P &= 0, \\
Q &= Ip \otimes C_m,
\end{align*}

The steady-state of \( e \) for \( u \equiv 0 \) is “matched” by the equivalent output disturbance generator:

\begin{align*}
\dot{x}_{ed}(t) &= A_{ed}x_{ed}(t) + \delta(t)L_{ed}(e(t) - \tilde{e}(t)), \\
\tilde{e}(t) &= C_{ed}x_{ed}(t), \\
L_m : \text{spec}(A_m - L_mC_m) &\subset \mathbb{C}_g, \text{[chosen for fast decay of the estimation error]} \\
\delta(t) &= \begin{cases} 
0, & \text{if } \max_{t\in[t-T,t]}|e(t) - \tilde{e}(t)| < \epsilon, \\
1, & \text{otherwise},
\end{cases}
\end{align*}

Generate the “(almost) disturbance free” output as \( \tilde{e} = e - \tilde{e} \)

---

Proposition

For any \( \epsilon > 0 \) there exist \( T_2^* > 0 \) such that \( T > T_2^* \) implies \( |e(t) - \tilde{e}(t)| < \epsilon \), for all \( t \geq T \).
Section 3

An example for the non hybrid case
A numerical example: data

Consider the plant described by the matrices:

\[
A = \begin{bmatrix}
-2 & 1 & 0 \\
-1 & -2 & 0 \\
-2 & 0 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 1 \\
1 & -1 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
-3 & 0 & -3 & 0 & -3
\end{bmatrix}, \quad Q = \begin{bmatrix}
2 & 0 & 1 & 0 & 1 \\
1 & -1 & 0 & 0 & 0
\end{bmatrix},
\]

The exosystem is characterized by \(\omega_0 = 0, \omega_1 = \frac{\pi}{4}, \omega_2 = \frac{3\pi}{5}\) and initial state \(q(0) = [-1 \ 0 \ 1 \ -1 \ 0]'.\)

**None of these plant data is used for control design**, except the values \(\omega_h, \ h = 0, 1, 2\).
A numerical example: practical regulation

Two Inputs, Two Outputs case with $\epsilon = 10^{-4}$.
Evolution of the two regulated outputs (top) and control inputs (bottom).
A numerical example: practical regulation

Single Input, Single Output case with $\varepsilon = 10^{-4}$.

Evolution of the first regulated output $e_1$ (top) and first control input $u_1$ (bottom).
A numerical example: practical regulation

Two Inputs, Single Output case with $\varepsilon = 10^{-4}$.
Evolution of the first regulated output $e_1$ (top) and two control inputs (bottom).
Smaller input amplitudes are used (with respect to the SISO case).
Asymptotic regulation via periodic resets with $\varepsilon = 0.5$.

Evolution of the regulated output $e$ (top) and control input $u$ (bottom).

The state of the exosystem is reset after 38 time units.
Conclusions

- Still lots of interesting (and fun!) things going on in **linear** regulation
- Data-driven external model based for hybrid output regulation: in CDC2016
- Hybrid output regulation still largely to understand even in very simple cases