Distributed optimization over networks: application to multi-building energy management

Maria Prandini
Politecnico di Milano, Italy
maria.prandini@polimi.it

Credit

Alessandro Falsone  Daniele Ioli  Simone Garatti  Kostas Margellos
Outline

- Building energy management: from a single building to a multi-building setup
- Distributed optimization over networks
- Distributed data-driven optimization over networks

Building cooling system with thermal storage
We consider a setup that comprises

- a building composed of a number of thermally controlled zones
We consider a setup that comprises
- a building composed of a number of thermally controlled zones
- a chiller plant that converts the electrical energy in cooling energy
- a thermal storage unit that accumulates/releases cooling energy and hence shifts in time the cooling energy request to the chiller plant

Objective:
- operate the building cooling system with thermal storage so as to guarantee a certain comfort when occupants are present, while minimizing the electrical energy cost
**Adopted approach**

Act on the temperature set-points of the zones and on the storage energy exchange

- **High-level controller**
- **Low-level Controlled Cooling System**
- **Storage**
- **Building**
- **Internal gains**
- **Temperature setpoint**
- **Storage contribution setpoint**
- **Environmental disturbances**
- **Cooling Energy**
- **Measured Zones Temperature**

**Optimal energy management**

Zone temperature set-points and storage charge/discharge command should be set appropriately in order to

- decrease the cooling power request
- exploit building thermal inertia to get an additional (passive) storage
- shift in time and set the cooling energy request to the chiller so as to use it at its maximal efficiency and request electrical energy to the grid when it is cheaper

while

- satisfying actuation constraints
- guaranteeing comfort conditions
Ingredients of the optimal control problem

- Cost function: electrical energy cost along some finite time-horizon
- Constraints:
  - comfort
  - actuation limits
- Control inputs:
  - zone temperature set-points \( u \)
  - thermal energy exchange with the storage \( s \)
- Disturbance inputs:
  - outdoor temperature
  - shortwave and longwave radiation
  - zone occupancy

Thermal energy balance

Cooling energy request to the chiller plant in each time slot \( dt \):

\[
E_{ch} = E_c - s
\]

- cooling load
- energy exchange with the storage
**Thermal energy balance**

Cooling energy request to the chiller plant in each time slot $dt$:

$$ E_{ch} = E_c - s $$

Corresponding electrical energy request:

$$ E_e = \alpha_1 T_o T_{cw} dt + \alpha_2 (T_o - T_{cw}) dt + \alpha_4 T_o E_{ch} \left\{ \frac{T_{cw} - \alpha_3 E_{ch}/dt}{T_{cw} - \alpha_3 E_{ch}/dt} \right\} - E_{ch} $$

- **$E_{ch}$**: Cooling load
- **$E_c$**: Energy exchange with the storage
- **$E_e$**: Corresponding electrical energy request
- **$T_o$**: Outdoor temperature
- **$T_{cw}$**: Temperature of the chilled water circuit, kept constant by low-level controller
Thermal energy balance

Cooling energy request to the chiller plant in each time slot $dt$:

$$E_{ch} = E_c - s$$

- cooling load
- energy exchange with the storage

Corresponding electrical energy request:

$$E_{el} = \alpha_1 T_o T_{cw} dt + \alpha_2 (T_o - T_{cw}) dt + \alpha_4 T_o E_{ch} - \frac{E_{ch}}{T_{cw} - \alpha_3 E_{ch}/dt}$$

$\rightarrow$ convex biquadratic approximation

$$E_{el} = c_1(T_o) E_{ch}^4 + c_2(T_o) E_{ch}^2 + c_3(T_o)$$

Efficiency of the chiller plant

Coefficient Of Performance:

$$\text{COP} = \frac{E_{ch}}{E_{el}}$$

![Graph showing COP vs. Energy Request (MJ)]
**Thermal energy balance**

Cooling energy request to the chiller plant in each time slot $dt$:

\[ E_{ch} = E_c - s \]

- **cooling load**
- **energy exchange with the storage**

Corresponding electrical energy request:

\[ E_\ell = c_1(T_o)E_{ch}^4 + c_2(T_o)E_{ch}^2 + c_3(T_o) \]

is convex in $E_{ch}$

Then, if $E_{ch}$ linear in $u$ e $s$

the cost function is convex in $u$ e $s$
Cooling energy request to the chiller plant in each time slot $dt$:

$$E_{ch} = E_c - s$$

energy exchange with the storage

cooling load

Cooling load:

$$E_o = \sum_{j=1}^{n_z} E_{c,j} = \sum_{j=1}^{n_z} (E_{w,j} + E_{p,j} + E_{int,j} + E_{z,j})$$

number of zones

time slot inertia

energy exchange

walls/zone

heat produced by people

heat produced by internal equipment, radiation through window

zone inertia
Cooling energy request to the chiller plant in each time slot $dt$:

$$E_{ch} = E_c - s$$

Cooling load:

$$E_c = \sum_{j=1}^{n_z} E_{c,j} = \sum_{j=1}^{n_z} (E_{w,j} + E_{p,j} + E_{int,j} + E_{z,j})$$

- Energy exchange with the storage
- Number of zones
- Energy exchange walls/zone
- Heat produced by people
- Heat produced by internal equipment, radiation through window
- Zone inertia

Thermal energy balance
Thermal energy balance

Cooling energy request to the chiller plant in each time slot $dt$:

$$E_{ch} = E_c - s$$

Corresponding electrical energy request:

$$E_e = c_1(T_o)E_{ch}^4 + c_2(T_o)E_{ch}^2 + c_3(T_o)$$

is convex in $E_{ch}$

Then, if $E_{ch}$ linear in $u$ e $s$

the cost function is convex in $u$ e $s$
Constraints

- Comfort constraint:
  zone temperature set-point belongs to some interval that may depend on the time slot
  \[ u_{min} \leq u \leq u_{max} \]

- Actuation constraints:
  - rate of charge/discharge of the storage
    \[ |s| \leq s_{max} \]
  - capacity of the storage
    \[ 0 \leq S \leq S_{max} \]
Constraints

- Comfort constraint:
  zone temperature set-point belongs to some interval that may depend on the time slot
  \( u_{min} \leq u \leq u_{max} \)

- Actuation constraints:
  - rate of charge/discharge of the storage
    \( |s| \leq s_{max} \)
  - capacity of the storage
    \( 0 \leq S \leq S_{max} \)
  - chiller plant cannot heat
    \( E_c \geq 0 \)
  - chiller plant saturation
    \( E_\ell \leq E_{max} \)

Constraints are convex in \( u \) and \( s \)
Constraints

- Comfort constraint:
  zone temperature set-point belongs to some interval that may depend on the time slot
  \[ u_{min} \leq u \leq u_{max} \] [linear in \( u \)]

- Actuation constraints:
  - rate of charge/discharge of the storage
    \[ |s| \leq s_{max} \] [linear in \( s \)]
  - capacity of the storage
    \[ 0 \leq S \leq S_{max} \]
  - chiller plant cannot heat
    \[ E_c \geq 0 \] [linear in \( u \)]
  - chiller plant saturation
    \[ E_t \leq E_{max} \] [convex in \( u \) and \( s \)]

Constraints

- Comfort constraint:
  zone temperature set-point belongs to some interval that may depend on the time slot
  \[ u_{min} \leq u \leq u_{max} \] [linear in \( u \)]

- Actuation constraints:
  - rate of charge/discharge of the storage
    \[ |s| \leq s_{max} \] [linear in \( s \)]
  - capacity of the storage
    \[ 0 \leq S \leq S_{max} \]
  - AR(1) model of the storage
    \[ S(k + 1) = aS(k) - s(k) \]
Constraints

- Comfort constraint:
  zone temperature set-point belongs to some interval that may depend on the time slot
  \[ u_{\text{min}} \leq u \leq u_{\text{max}} \quad \text{[linear in } u]\]

- Actuation constraints:
  - rate of charge/discharge of the storage
    \[ |s| \leq s_{\text{max}} \quad \text{[linear in } s]\]
  - capacity of the storage
    \[ 0 \leq S \leq S_{\text{max}} \quad \text{[linear in } s]\]
  - AR(1) model of the storage
    \[ S(k+1) = aS(k) - s(k) \]

Constraints are convex in \( u \) and \( s \)
Convex constrained optimization

The optimal energy management problem reduces to the following convex constrained optimization problem:

\[
\min_{u, s} \quad E_e(u, s)
\]

subject to:

\[
u_{\text{min}} \leq u \leq u_{\text{max}}, \quad |s| \leq s_{\text{max}}
\]

\[
0 \leq S \leq S_{\text{max}}, \quad E_c \geq 0, \quad E_e(u, s) \leq E_{\text{max}}
\]

A numerical example

Building structure

[Diagram of a building structure with labeled zones, surfaces, and dimensions]
A numerical example

Comfort constraints and energy price along a 1 day time-horizon

Disturbances along a 1 day time-horizon
Look-ahead time horizon: 24 hours
Time slot $dt$: 10 minutes

4 policies are compared:

- **Fixed:**
  - temperature kept constant during working hours; chiller idle otherwise
  - storage is charged at night and discharged during the day

- **Optimal:**
  - solution of the constrained optimization problem over 48 hours

- **Fixed without storage**

- **Optimal without storage**

### A numerical example: single zone

**Zone temperature set-point**

![Graph showing temperature over time for different policies](image-url)
A numerical example: single zone

Pre-cooling phase to exploit the building as a passive storage

![Graph showing temperature over time for different strategies.]

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
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<td>F</td>
<td>1094</td>
<td>1094</td>
<td>1219</td>
<td>29.09</td>
</tr>
<tr>
<td>F+S</td>
<td>1087</td>
<td>1330</td>
<td>742.3</td>
<td>16.95</td>
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<tr>
<td>O</td>
<td>1288</td>
<td>1288</td>
<td>750.5</td>
<td>16.63</td>
</tr>
<tr>
<td>O+S</td>
<td>1076</td>
<td>1187</td>
<td>694.4</td>
<td>14.44</td>
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</table>
The chiller plant works at better efficiency levels

<table>
<thead>
<tr>
<th>Strategy</th>
<th>F</th>
<th>F+S</th>
<th>O</th>
<th>O+S</th>
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</thead>
<tbody>
<tr>
<td>$E_c$ [MJ]</td>
<td>1094</td>
<td>1087</td>
<td>1288</td>
<td>1076</td>
</tr>
<tr>
<td>$E_{ch}$ [MJ]</td>
<td>1094</td>
<td>1390</td>
<td>1288</td>
<td>1187</td>
</tr>
<tr>
<td>$E_d$ [MJ]</td>
<td>1219</td>
<td>742.3</td>
<td>750.5</td>
<td>694.4</td>
</tr>
<tr>
<td>Cost [euro]</td>
<td>29.09</td>
<td>16.95</td>
<td>16.63</td>
<td>14.44</td>
</tr>
</tbody>
</table>

A numerical example: single zone

A numerical example: multiple zone setting

Zone temperature set-point is different for the three floors
A numerical example: multiple zone setting

Zone temperature set-point is different for the three floors

Optimal policy without storage is considered

A numerical example: multiple zone setting

Zone temperature set-points [optimal policy without storage]

The intermediate floor (zone 2) is used as a thermal storage which drains heat from other floors through its pavement and its ceiling.
A numerical example: single vs multiple zone

<table>
<thead>
<tr>
<th></th>
<th>Single-Zone</th>
<th>Multi-Zone</th>
<th>Δ%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_C$ [MJ]</td>
<td>1288</td>
<td>979.9</td>
<td>-23.9%</td>
</tr>
<tr>
<td>$E_{ch}$ [MJ]</td>
<td>750.5</td>
<td>669.7</td>
<td>-14.5%</td>
</tr>
<tr>
<td>Cost [euro]</td>
<td>16.63</td>
<td>12.79</td>
<td>-21.1%</td>
</tr>
</tbody>
</table>

The added flexibility allows to better exploit the building inertia.

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**Single building setup**

- a convex formulation of the optimal energy management problem for a building cooling system with thermal storage was introduced
- the model is easily scalable in the number of zones, can be generalized to a multi-building setup
Multi-building setup

District network

- $m$ buildings, possibly divided into zones
- a chiller unit for each building
- a single shared energy storage

- Control Inputs:
  - zone temperatures
  - storage energy exchange

- Disturbance inputs:
  - occupancy
  - outdoor temperature
  - shortwave and longwave radiation
Multi-building setup

District network
- $m$ buildings, possibly divided into zones
- a chiller unit for each building
- a single shared energy storage

Control Inputs:
- zone temperatures
- storage energy exchange

Disturbance inputs:
- occupancy
- outdoor temperature
- shortwave and longwave radiation

Issues
- Computation: Problem size too big!
- Communication: Not all communication links at place; link failures
- Information privacy: buildings may not want to share their consumption profiles

Outline
- Building energy management problem: from a single building to a multi-building setup
- Distributed optimization over networks
- Distributed data-driven optimization over networks
Problem setup

- network of $i = 1, 2, \ldots, m$ cooperative agents
- let $x$ denote the global decision vector to be optimally agreed upon
Centralized optimization problem

\[
\min_x \sum_{i=1}^{m} f_i(x) \\
\text{subject to } x \in \bigcap_{i=1}^{m} X_i
\]

- \( f_i(x) \): Objective/utility function of agent \( i \)
- \( X_i \): Physical/technological constraints of agent \( i \)
- Coupling via common decision variables and constraints

Distributed architecture

**Step 1:** agent \( i \) solves a local decision problem and makes a tentative (local) decision for \( x \)
Step 2: neighbouring agents communicate their tentative decisions to agent $i$.

Step 3: Agent $i$ weights the received information, solves a refined problem and makes a new decision for $x$. 
1: Initialization
2: \( k = 0 \).
3: Consider \( x_i(0) \in X_i \), for all \( i = 1, \ldots, m \).
4: For \( i = 1, \ldots, m \) repeat until convergence
5: \( z_i(k) = \sum_{j=1}^{m} a_j^i(k) x_j(k) \).
6: \( x_i(k+1) = \arg \min_{x_i} \left( f_i(x_i) + \frac{1}{2\varepsilon(k)} \| z_i(k) - x_i \|^2 \right) \).
7: \( k \leftarrow k + 1 \).
Scalable method:
- communication only between neighbours
- computation only local, in parallel for agents

Information privacy:
- agents do not share information about their preferences/needs with the other agents

---

Convergence analysis

- $f_i(x)$: convex
- $X_i$: convex, compact

$$\min_{x_i \in X_i} f_i(x_i)$$
Convergence analysis

- $f_i(x)$: convex
- $X_i$: convex, compact

$$\min_{x_i \in X_i} f_i(x_i) + \frac{1}{2c(k)} ||z_i(k) - x_i||^2$$

Choice of the proxy term

Penalty coefficient $c(k)$ is positive, non-increasing, should not decrease too fast

$$\sum_{k=0}^{\infty} c(k) = \infty,$$
$$\sum_{k=0}^{\infty} c(k)^2 < \infty.$$

so that asymptotically we give emphasis to consensus, but without compromising the possibility to achieve optimality

Information mix

Weight coefficients in $z_i(k) = \sum_{j=1}^{m} a^i_j(k)x_j(k)$

- Satisfy a non-zero lower bound if link $i$-$j$ is present
  → info mixing at a non-diminishing rate

- form a doubly stochastic matrix, i.e.
  $$\sum_{i=1}^{m} a^i_j(k) = 1$$
  $$\sum_{j=1}^{m} a^i_j(k) = 1$$
  → agents influence each other equally in the long run
Convergence analysis

Network connectivity

- Any pair of agents communicates infinitely often, possibly through a communication graph that changes through iterations

- The intercommunication time is bounded

Convergence analysis

- Consensus: agents’ estimates converge to their arithmetic average

\[ x_i(k) \xrightarrow{k \to \infty} \frac{1}{m} \sum_{i=1}^{m} x_i(k) \]
Convergence analysis

- Consensus: agents’ estimates converge to their arithmetic average

\[ x_k(k) \xrightarrow[k \to \infty]{} \frac{1}{m} \sum_{i=1}^{m} x_i(k) \]

- Optimality: asymptotic convergence to some minimizer of the centralized problem

\[ x_k(k) \xrightarrow[k \to \infty]{} x^* \]

Numerical example – District definition

District configuration
3 identical buildings with three zones
and with different chillers

Chiller types:
‘small’ → building 2
‘medium’ → building 1
‘large’ → building 3
Results: zone temperature set-points

- Strong pre-cooling phase
- Middle floor (green line) used as additional thermal storage

Energy exchange with the storage: solution computed by building 1
Realizations of the disturbances along a 1 day time-horizon

Solar radiation data

Courtesy of Istituto di Scienze dell'Atmosfera e del Clima (ISAC) - CNR
Solar radiation data

Bologna (ISAC) 44.5N 11.3E - Real Time Radiation (15 min update) 20110613

SW Global
SW diffuse
SW DIF\(\times\)CosZ
SW DIF
LWIR
LWIR(w=1)
CosZ\(^{10000}\)

0 3 6 9 12 15 18 21 24
GTM Time (h)

In irradiance (W/m²)

2.7 kWh/m²

Courtesy of Istituto di Scienze dell'Atmosfera e del Clima (ISAC) - CNR

Solar radiation data

Bologna (ISAC) 44.5N 11.3E - Real Time Radiation (15 min update) 20110727

SW Global
SW diffuse
SW DIF\(\times\)CosZ
SW DIF
LWIR
LWIR(w=1)
CosZ\(^{10000}\)

0 3 6 9 12 15 18 21 24
GTM Time (h)

In irradiance (W/m²)

3.3 kWh/m²

Courtesy of Istituto di Scienze dell'Atmosfera e del Clima (ISAC) - CNR
Building energy management problem: from a single building to a multi-building setup

Distributed optimization over networks

Distributed data-driven optimization over networks

\[ \min_x \sum_{i=1}^{m} f_i(x) \]
subject to \( x \in \bigcap_{i=1}^{m} X_i \)

\[ X_i \rightarrow X_i(\delta) \]

\( \delta \): uncertainty vector
Centralized stochastic optimization

\[
\min_x \sum_{i=1}^{m} f_i(x) \\
\text{subject to } x \in \bigcap_{i=1}^{m} X_i
\]

\[
X_i \rightarrow X_i(\delta)
\]

- $\delta$: uncertainty vector
- uncertainty distributed on set $\Delta$ according to probability $P$
- only scenarios $\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)}$ available
Centralized stochastic optimization

\[ \min_{x} \sum_{i=1}^{m} f_i(x) \quad \rightarrow \quad x_N^c \]

subject to \( x \in \bigcap_{k=1}^{N} \bigcap_{i=1}^{m} X_i(\delta^{(k)}) \)

how does the solution \( x_N^c \) generalize to unseen scenarios?
Centralized stochastic optimization

\[ \begin{align*}
\min_x & \sum_{i=1}^{m} f_i(x) \\
\text{subject to } & x \in \bigcap_{k=1}^{N} \bigcap_{i=1}^{m} X_i(\delta^{(k)}) \end{align*} \]

how does the solution \( x^c_N \) generalize to unseen scenarios?

\[ \mathbb{P} \left\{ \delta : x^c_N \notin \bigcap_{i=1}^{m} X_i(\delta) \right\} \]

Scenario approach

\[ \begin{align*}
\min_{(\eta, h)} & h \\
\text{subject to } & l_\delta(\eta) \leq h, \forall \delta \in \Delta
\end{align*} \]

Uncertainty set \( \Delta \) endowed with a probability measure \( \mathbb{P} \)
Scenario approach

Pick $\delta^{(1)}$, $\delta^{(2)}$, ..., $\delta^{(N)}$ at random from $\Delta$, according to $P$

\[ \min_{\eta, h} h \]

subject to $l_{\delta^{(i)}}(\eta) \leq h, i = 1, ..., N$

Consider only a finite number $N$ of constraints
Scenario approach

Pick $\delta^{(1)}$, $\delta^{(2)}$, ..., $\delta^{(N)}$ at random from $\Delta$, according to $P$.

Consider only a finite number $N$ of constraints.

$\min h$

subject to $l_{\delta^i}(\eta) \leq h$, $i = 1, ..., N$

how robust is the scenario solution $\theta_N = (\eta_N, h_N)$?

Scenario approach

$\{\delta: l_{\delta}(\eta_N) > h_N\}$ is the Violation Set of $\theta_N = (\eta_N, h_N)$

$V(\theta_N) = P\{\delta: l_{\delta}(\eta_N) > h_N\}$ is the Violation of $\theta_N = (\eta_N, h_N)$. 

Scenario approach

\( \{ \delta : l_\delta(\eta_N) > h_N \} \) is the Violation Set of \( \theta_N = (\eta_N, h_N) \)

\( V(\theta_N) = P\{ \delta : l_\delta(\eta_N) > h_N \} \) is the Violation of \( \theta_N = (\eta_N, h_N) \)
The violation $V(\theta_N)$ is a random variable with probability distribution

$$F_V(\varepsilon) := P_N \{ V(\theta_N) \leq \varepsilon \}$$
Scenario approach

\[ \min_{(\eta, h)} h \]
\[ \text{subject to } l_{\delta(i)}(\eta) \leq h, i = 1, ..., N \]

Theorem

If \( f_\delta(\theta) = l_\delta(\eta) - h \) is convex in \( \theta = (\eta, h) \in \mathcal{R}^d \), then

\[ P^N \{ V(\theta_N) \leq \varepsilon \} \geq 1 - \beta = 1 - \sum_{i=0}^{d-1} \binom{N}{i} \varepsilon^i (1 - \varepsilon)^{N-i} \]

Scenario approach

\[ \min_{(\eta, h)} h \]
\[ \text{subject to } l_{\delta(i)}(\eta) \leq h, i = 1, ..., N \]

Take \( \beta \), say \( \beta = 10^{-7} \). Then,

1. \( N \geq \frac{1}{\varepsilon} \left( d + \log \left( \frac{1}{\beta} \right) + \sqrt{2d \log \left( \frac{1}{\beta} \right)} \right) \) to get a violation \( \varepsilon \)
Scenario approach

\[
\begin{align*}
\min_{(\eta, h)} h \\
\text{subject to } l_\delta(i)(\eta) \leq h, i = 1, ..., N
\end{align*}
\]

Take \( \beta \), say \( \beta = 10^{-7} \). Then,

- \( N \geq \frac{1}{\varepsilon} \left( d + \log \left( \frac{1}{\beta} \right) + \sqrt{2d \log \left( \frac{1}{\beta} \right)} \right) \) to get a violation \( \varepsilon \)

- If \( \tilde{N} \) realizations are available (data driven problem), then

\[
P^N \{ V(\theta_N) \leq \varepsilon \} \geq 1 - \beta \quad \text{where } \varepsilon = 1 - \tilde{N}^{-d} \sqrt[2d]{\frac{\beta}{\tilde{N}^d}}
\]

Scenario approach

\[
\begin{align*}
\min_{(\eta, h)} h \\
\text{subject to } l_\delta(i)(\eta) \leq h, i = 1, ..., N
\end{align*}
\]

Theorem

If \( f_\delta(\theta) = l_\delta(\eta) - h \) is convex in \( \theta = (\eta, h) \in \mathcal{R}^d \), then

\[
P^N \{ V(\theta_N) \leq \varepsilon \} \geq 1 - \beta = 1 - \sum_{i=0}^{d-1} \binom{N}{i} \varepsilon^i (1 - \varepsilon)^{N-i}
\]

upper bound on the cardinality of the support set
**Scenario approach**

**support set**

“minimal cardinality subset of the constraints such that by considering only this set of constraints, we obtain the same solution”

**Intuition:**
all constraints that do not belong to the support set are in a sense redundant → generalization to unseen scenarios

---

**Scenario approach: a posteriori result**

Let \( \hat{d} \) be the cardinality of the support set for the extracted scenario program, then,

\[
P_N \{ V(\theta_N) \leq \hat{\epsilon}(\hat{d}) \} \geq 1 - \beta \text{ where } \hat{\epsilon}(k) = 1 - \frac{N-k}{\sqrt{(\hat{d}+1)\binom{N}{k}}}\beta
\]

**Remarks:**
- also for non-convex problems
- ‘wait and see’ a-posteriori result
Centralized stochastic optimization

\[
\min_x \sum_{i=1}^{m} f_i(x) \quad \rightarrow \quad x^c_N
\]

subject to \( x \in \bigcap_{k=1}^{N} \bigcap_{i=1}^{m} X_i(\delta^{(k)}) \)

how does the solution \( x^c_N \) generalize to unseen scenarios?

\[
P \left\{ \delta : x^c_N \notin \bigcap_{i=1}^{m} X_i(\delta) \right\}
\]

\[
\text{known upper bound for the support set cardinality}
\]
### Centralized stochastic optimization

\[
\min_x \sum_{i=1}^m f_i(x)
\]
subject to \( x \in \bigcap_{k=1}^N \bigcap_{i=1}^m X_i(\delta^{(k)}) \)

- with confidence \( 1 - \beta \) it holds that
\[
\mathbb{P} \left\{ \delta : x_N^c \notin \bigcap_{i=1}^m X_i(\delta) \right\} \leq \varepsilon = 1 - \sqrt{\frac{N \beta}{d}}
\]
- \( x_N^c \) can be computed via the proposed distributed algorithm, but scenarios should be available to all agents

### Distributed stochastic optimization

- Scenarios (constraints) are private resources
- Each agent has its own \( N_i \) scenarios/realizations of \( \delta \)
\[
\delta^{(i,1)}, \delta^{(i,2)}, \ldots, \delta^{(i,N_i)}
\]
Distributed stochastic optimization

- Scenarios (constraints) are private resources
- Each agent has its own $N_i$ scenarios/realizations of $\delta$
  
  $\delta(i,1), \delta(i,2), \ldots, \delta(i,N_i)$

Construct the scenario program

$$\min_x \sum_{i=1}^{m} f_i(x)$$

subject to $x \in \bigcap_{i=1}^{m} \bigcap_{k=1}^{N_i} X_i(\delta(i,k))$

Distributed stochastic optimization

$$\min_x \sum_{i=1}^{m} f_i(x) \quad \rightarrow \quad x_N^d$$

subject to $x \in \bigcap_{i=1}^{m} \bigcap_{k=1}^{N_i} X_i(\delta(i,k))$

- fits the distributed set-up with
  $\bigcap_{k=1}^{N_i} X_i(\delta(i,k))$ in place of $X_i$
- $x_N^d$ can be computed via the proposed distributed algorithm
Distributed stochastic optimization

\[
\min_x \sum_{i=1}^{m} f_i(x) \quad \Rightarrow \quad x_d^N
\]

subject to \( x \in \bigcap_{i=1}^{m} \left[ \bigcap_{k=1}^{N_i} X_i(\delta^{(i,k)}) \right] \)

\[ \mathbb{P} \left\{ \delta : x_d^N \notin \bigcap_{i=1}^{m} X_i(\delta) \right\} \leq ? \]

solution to the distributed problem,
with local scenarios
Distributed stochastic optimization

\[
\min_x \sum_{i=1}^m f_i(x) \quad \rightarrow \quad x^d_N
\]

subject to \( x \in \bigcap_{i=1}^m \bigcap_{k=1}^{N_i} X_i(\delta^{(i,k)}) \)

\[
\mathbb{P} \left\{ \delta : x^d_N \notin \bigcap_{i=1}^m X_i(\delta) \right\} \leq ?
\]

solution to the distributed problem, with local scenarios

constraint of the centralized problem, with global scenarios

Distributed stochastic optimization

\[
\min_x \sum_{i=1}^m f_i(x) \quad \rightarrow \quad x^d_N
\]

subject to \( x \in \bigcap_{k=1}^N \bigcap_{i=1}^m X_i(\delta^{(k)}) \)

**Theorem (extensions of scenario theory to distributed optimization)**

Fix \( \beta \) and choose \( \beta_i \) such that \( \sum_{i=1}^m \beta_i = \beta \). Then, with confidence 1-\( \beta \) it holds that

\[
\mathbb{P} \left\{ \delta : x^d_N \notin \bigcap_{i=1}^m X_i(\delta) \right\} \leq \sum_{i=1}^m \varepsilon_i(\hat{d}_i)
\]

a-posteriori bound
**Distributed stochastic optimization**

\[
\min_{x} \sum_{i=1}^{m} f_i(x) \quad \rightarrow \quad x_N^d
\]

subject to \( x \in \bigcap_{k=1}^{N} \bigcap_{i=1}^{m} X_i(\delta^{(k)}) \)

**Theorem (extensions of scenario theory to distributed optimization)**

Fix \( \beta \) and choose \( \beta_i \) such that \( \sum_{i=1}^{m} \beta_i = \beta \). Then, with confidence \( 1-\beta \) it holds that

\[
\mathbb{P} \left\{ \delta : x_N^d \notin \bigcap_{i=1}^{m} X_i(\delta) \right\} \leq \sum_{i=1}^{m} \varepsilon_i(d_i) \leq \varepsilon = \max_{\{d_i \in \mathbb{R}_+\}} \sum_{i=1}^{m} \varepsilon_i(d_i)
\]

subject to \( \sum_{i=1}^{m} d_i \leq d \)

**Conclusions**

- New results on distributed data-driven optimization
- Extension of the scenario approach to distributed convex optimization
- Still much work need to be done to quantify achievable performance and to obtain receding horizon implementation
Uncertainty description “a-priori”

30 realizations of the energy produced by a photovoltaic panel installation
[courtesy of GE Global Research Europe, Munich]

Receding horizon implementation
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Thank you for your attention!